BROWN

BROWN UNIVERSITY

COMPUTER SCIENCE DEPARTMENT

AN ALGORITHMIC THEORY OF GENERAL EQUILIBRIUM

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M.S. IN COMPUTER SCIENCE, BROWN UNIVERSITY, 2023

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THESIS SUBMITTED IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF DOCTOR OF PHILOSOPHY.

KEY WORDS: GAME THEORY, VARIATIONAL INEQUALITIES, GENERAL EQUILIBRIUM THEORY, MULTIAGENT LEARNING, OPTIMIZATION, COMPUTATIONAL COMPLEXITY, DEEP LEARNING

I pursued this thesis as one pursues a work of art.

Dedicated to my mother Deniz
whose love turns the heartlines in my hands into golden threads of fate.
Thank you for everything.

ABSTRACT

In the wake of the Second World War, as they presided over major public expenditure reforms, European and American governments supported the development of rigorous mathematical models of economies to guide economic policy. Over the next two decades, general equilibrium models (or Walrasian economies) emerged as the dominant framework. However, as these models were often analytically intractable, as early as the 1960s, a group of researchers led by Herbert Scarf turned their attention to finding "a general method for the explicit numerical solution of [general equilibrium models]." While some methods had limited success in solving simple models, 50 years later, a general method for computing solutions to more complex models remains elusive. Nevertheless, these models—and their often inaccurate solution methods—continue to be widely used in applications such as resource allocation and public policy analysis, raising concerns about the impact of inaccurate solutions on the public good. This thesis addresses this issue by leveraging tools from computer science and game theory to analyze algorithms for general equilibrium models.

The first part of this thesis focuses on variational inequalities (VIs), a mathematical modeling paradigm, and their application to Walrasian economies (i.e., models driven by demand and supply). I introduce a new family of polynomial-time first-order methods for solving VIs that satisfy the Minty condition and prove that any solution of a balanced economy—a large class that includes the seminal Arrow-Debreu economies—can be cast as the solution of a VI satisfying the Minty condition. Using this framework, I develop the first globally convergent family of polynomial-time price-adjustment methods for solving balanced Walrasian economies, thereby resolving the half-century-old challenge set by Scarf. Additionally, for the special class of Walrasian economies known as homothetic Fisher markets, I prove the convergence of the widely used *tâtonnement* process. Beyond balanced Walrasian economies, I also develop globally convergent polynomial-time merit function methods for Walrasian economies under mild smoothness assumptions.

The second part of this thesis concerns pseudo-games, a multiagent optimization framework, and their application to Arrow-Debreu economies (i.e., Walrasian economies in which demand and supply are generated by consumers and firms). I introduce a new family of polynomial-time learning dynamics for solving variationally stable pseudo-games—a class that includes (quasi)monotone pseudo-games—and prove that any solution of a pure exchange economy (i.e., an Arrow-Debreu economy without firms) can be cast as a solu-

tion of a variationally stable pseudo-game. This result provides a new family of globally convergent polynomial-time market dynamics for solving pure exchange economies and offers an alternative resolution to the challenge set by Scarf. Finally, extending beyond pure exchange economies, I develop globally convergent polynomial-time merit function methods for Arrow-Debreu economies under mild smoothness assumptions.

The final part of this thesis concerns Markov pseudo-games, a multiagent *stochastic* optimization framework, and their application to Radner economies (i.e., a generalization of Walrasian and Arrow-Debreu economies that explicitly model time and uncertainty). Since solutions of Markov pseudo-games are generally infinite-dimensional, I focus on solving these models using function approximation methods inspired by merit functions (e.g., deep learning) and develop a polynomial-time globally convergent method under suitable assumptions on the Markov pseudo-game. I then show that the solution of any Radner economy can be cast as a solution of a Markov pseudo-game, thereby providing a general solution method with convergence guarantees for Radner economies. While Parts 1 and 2 of this thesis resolve Scarf's challenge by solving general equilibrium models developed during his lifetime, Part 3 merely scratches the surface of a new research direction. Above all, it raises an exciting and contemporary analog to Scarf's challenge at the intersection of deep learning, reinforcement learning, and mathematical economics—namely, finding a general method for the explicit numerical solution of Radner economies.

Acknowledgments

One thing my PhD never lacked was adventure. My journey started in the midst of a global pandemic and took me from San Francisco to Berlin, Vienna, London, and Paris, giving me the privilege of spending time with brilliant researchers at the Simons Institute, Google DeepMind, and JP Morgan.

The architect of these opportunities was, without a doubt, my advisor Amy Greenwald. Amy, the woman you are! Compliments would not do you justice, so I'll spare you them. During my first year, the day after a dramatic encounter in the CIT elevator, you said, "Perhaps it's not a coincidence that they call the PhD defense a *viva* in the UK—your task is to defend your research with your life!" I carried those words with me for the rest of my PhD. The only word I can find to describe the last six years together is *wild*, and I am sure we would not have it any other way. Thank you for listening to me and for caring about what I had to say; it is the greatest gift one can receive. I will forever carry the memories and lessons of our almost religious, hours-long conversations on game theory; our long walks through New York, Berkeley, London, New Haven, and Providence; you grilling me in our reading group meetings; and the exacting paper submission deadlines we faced together. *At the start of my PhD, I had one idea of what the advisor-advisee relationship is and should be, now at the end of it, I see that I got that and everything more. Thank you, thank you for everything.*

The inspiration behind my first research projects on general equilibrium were the works of Richard Cole, Yun Kuen (Marco) Cheung, Simina Brânzei, and Christian Kroer. Few people remain in the world with the attention span to catch a mathematical error in Equation 132 of a 50-page paper; Richard and Marco are among them. Thank you for being the trailblazers, giving me endless feedback, and inspiring me to be a better researcher. I wrote every single word in this thesis thinking, "what would Richard think"? Simina, thanks for the numerous chats at Berkeley, hearing your stories, and seeing your passion for research take you all around the world is an inspiration. Christian, thank you for spending time on my committee, we did not get a chance to work closely, but knowing how highly your students speak of you, it speaks volumes to how we should have. At this point, I should also acknowledge two other committee members who have generously joined the committee for this thesis, Yu Cheng and Geoffrey Gordon. Thank you for your time, efforts, and interest in my growth, I am grateful.

I would not have applied to a PhD program if it weren't for two people: Seth Goldstein and Ryan Zohar. Seth, in many ways, this thesis was written to answer the very questions we sought to explore in that first-ever research project we started together. Thank you for inspiring me—without you, I would never have had the chance to write it, and for that, I am deeply grateful. Ryan, my curious friend, your passion for learning, your desire for discovery, and your encouragement to apply for the PhD are why I pursued it in the first place. I cannot wait to see where life takes us. I love you.

One of the most exciting opportunities in my life was working at DeepMind in London right after the pandemic. I owe this opportunity to two people: Andrea Tacchetti and Kalesha Bullard. Andrea, thank you for giving me the chance to work with you—I am forever indebted. Your kindness, patience, brilliance, and resilience in expanding into new research horizons, whether you realize it or not, have taught me invaluable lessons that I carry with me. *Thank you*. Kalesha, thank you for preparing me for my interviews, for the time spent together in London's restaurants and markets, and for sharing your stories with me. You are a treasure. You must know that Minae and I refer to you as *the icon*, and I know you know you are one.

Once I arrived at DeepMind, an incredibly talented group of researchers became my champions: David Parkes, Guy Lever, Ian Gemp, Georgios Piliouras, Luke Marris, Yoram Bachrach, Romuald Élie, Minae Kwon, Robert Lange, Evgenii Nikishin, and Chris Lu. David, thank you for supporting my growth and taking an interest in my research. But most importantly, thank you for being a resilient role model—you should know it means the world to me. Guy, my rock, your sense of humor, your perspective on the world, and your steadfast sense of identity made me feel at home at DeepMind. Thank you. Ian, of all the researchers I have met, you are the one with the purest and most genuine love for research. Learning from you and sharing ideas with you was an immense and inspiring experience. Thank you. Georgios, your research creativity, and desire to explore topics at the edge of the field has inspired much of this thesis starting from Berkeley all the way to London, thank you for sharing your ideas and inspirations with me. Luke, you taught me so much about machine learning engineering—thank you for your tireless help. Yoram and Romuald, thank you both for the many hours spent chatting with me, pointing me toward exciting research directions, and supporting me at DeepMind. Minae, you brilliant, noble, and extraordinary woman, I am so grateful to know you and call you my friend. Robert, you loving man, in another life, I know we were together. Evgenii, dreamer, keep sailing toward new horizons—you will only continue to blossom. Chris, kind soul, I am excited to see the great things you will build.

In London, I spent time with three old friends and three new special people. Deniz Elif, my strong and loving friend, *I adore you*. Will, we keep changing, but I know our friendship will always remain the same. *I love you.* Rostom, dostum, thanks for helping me out during my move to London, I am so grateful for it, and know that I am always here for you too if you ever need help. I hope our paths cross again. Rif and David—Paris, London, or New York, I know we will find each other. Domenico, the time, conversations, and meals we shared in my final months in London meant the world to me. Thank you for those moments—*you have a special place in my heart forever*.

Once back in Providence from London, two talented and resilient researchers from Caltech reached out to me with a question about a paper not included in this thesis: Apurva Badithela and Josefine Graebener. This led to a two-year collaboration on a fascinating project using game theory to solve automated test design. Apurva and Josefine, thank you for sharing your ideas with me and for teaching me about your field.

Before leaving Providence for London to join DeepMind, I was awarded a JP Morgan AI fellowship and put in touch with Sumitra Ganesh to be her mentee. Back from DeepMind, I was hoping to do research at a place with a focus on economics. So, I reached out to Sumitra to ask if I could join her team of researchers for the summer, and she kindly agreed. Sumitra, thank you for giving me research freedom and supporting my research, I am grateful. At JP Morgan, my greatest champion was my research mentor Alec Koppel. Alec, you and the world, knows you are a brilliant and prolific researcher, but to me what mattered was your love for life, your support for those you mentor, and your unbeatable small-talk game. The paper we wrote at JP Morgan was not included in this already far-too-long thesis, but it was a paper I had dreamed of writing for a very long time, and I believe it is one of the most important papers I published in my PhD. *Thank you for trusting my vision, and supporting this research with no hesitation even in the face of setbacks, you are a role-model.* At JP Morgan, I worked with other brilliant researcher: Sujay Bhatt, Alan Mishler, Kevin Stanghl, Thomas Cook. Sujay and Allan thank you for welcoming me, and making me feel at home. Kevin, my friend at first sight, you are brilliant, kind, and genuine, trust yourself you are made for success. Tom, my big-hearted Polish man, getting to hear your stories was a treat, never doubt yourself.

Once I left JP Morgan, I set my sights on accomplishing the dream I entered grad school with: building economic simulations with general equilibrium theory. With the goal to find the right literature, I learned more about macroeconomics and stumbled on the works of two great economists: Jesse Perla and Adrien Auclert. Jesse, thank you for caring, being open to listen to and read my ideas. It meant a lot to me at a moment where I felt all alone trying to make sense of the literature. Adrien, thank you for giving me the opportunity to participate in your NBER workshop on Heterogeneous-Agent Macroeconomics, I know non-economists rarely get the chance to do so. Learning about your research and communities was pivotal in concluding the final part of this thesis. I am hopeful that in the near future, computer scientists and economists, will join their forces to solve pressing macroeconomic problems.

In the midst of the never-ending cycle of paper submissions, I was fortunate to meet a brilliant and unique student who has become my greatest supporter and research partner, Sadie Zhao. Sadie, I cannot believe it has been four years since we met! Seeing you grow from an undergraduate interested in research to a PhD student with a research agenda, and now a friend, has been such a joy. *Your future is bright, dream on, and reach for the stars!* The last part of this thesis was a collaboration with Sadie, Amy, and most importantly Sadie's PhD advisor Yiling Chen. Yiling, thank you for trusting our vision with Sadie and supporting this research; this was one of the most challenging research projects I have undertaken, and your support meant so much to me. I am so happy to know Sadie is getting advised by someone she can look up to and get inspired by.

In my years at Brown, I was lucky to spend time with a number of PhD students: Brandon Woodard, Enrique Areyan, Arjun Prakash, and Kevin Wang. Brandon, thank you for the memories and the experiences, I cannot wait to see the things you will do. Enrique, thank you for helping me transition to Brown. Arjun, thank you for the countless hours we have spent working together, I am excited to see the great places life takes you. Kevin, I only wish we could have spent more time, but I trust our paths will cross again.

There was also a very special PhD student: Lucy Qin. Lucy, my kind, beautiful, free, and brilliant partner in crime. I do not know how I would have done this PhD without you. You have been with me on my hardest and happiest days, from Fox Point, Providence, to Chinatown, New York, to Miami Beach, Miami. People like you, and a friendship like ours come once in a lifetime. You have been one of the greatest finds of this journey. *I love you.*

Beyond the support I have received from people I met during my PhD, I am eternally grateful for the support of three groups of people. Chief amongst them, my friends: Eric Deng, Roysi Eskenazi, Timur Emre Civan, Audrey Everist, Lisa Inzerillo, Sebastian Torero, Rebecca Vitenzon, Rostom Bouazizi. Eric, your resilience, love, loyalty, and wisdom beyond your years, has been one of the best gifts of my life. How is it possible to not have a single bad memory in seven years of friendship? I have written this thesis with your number one lesson in mind "Diamonds are made under pressure". From the long nights and mornings of Unter, to the sunny days of Jacob Riis and Pier 45, it is with you that I discovered who I am. In the first very hard years of my PhD the time I would spend with you in New York, meant more than the world to me. One early winter morning in the middle of nowhere in Brooklyn, you said "We keep on experiencing things people in the movies can only dream of", and I know that there are many more of these moments on the horizon, and I cannot wait to share them with you. If you wanted to, you could rule the world, but you choose not to. *I love you.* Roysi, my gentle, loving, open-minded and perfectionist friend. You have expanded my world in ways you do not even know, made me at home with your open heart and arms. In the most isolating and dark days of the pandemic, the long hours we spent on the phone together were perhaps the one thing that kept me going through this PhD. I know we will keep on growing, but never apart. *I love you.* Timur, my kind-hearted and dependable friend, how will I ever pay you back for all the help you have given me when I was first moving to Providence? I cannot wait to see the beautiful life you will build for yourself in London, and while I miss you every day, know that I will always be there for you. We will keep on sailing to new horizons until we find our home. *I love you.* Audrey, my freedom-loving American girl, you are a dying breed. Your bravery, joy, wit, and curiosity for the world keep on inspiring me. Keep running, do not stop for a normal life, we will eventually build the extraordinary together, "normal" is not meant for girls like us. I cannot wait for the laughs we will share in the years to come. *I love you*. Lisa, *mon coeur, être une femme libérée ce n'est pas facile*, but you sure make it seem so. As we try to figure out who we want to become, I know you will always choose to be yourself, and in doing so, you will never be wrong. The hardships will come and go, and we will keep on flourishing. *I love you.* Sebastian, my darling bunkmate, your patience and compassion encourage me to be a better person. I do not know if you know this but in the loneliest days of the pandemic, your mother sent me a care package; that small act of kindness kept me going. So, to thank your Mother who once said "They're trying a bunch of stuff to lower inflation rather than just lower it", I wrote this thesis with the dream that one day, "they" stop trying a bunch of stuff and instead just "lower" it. Maybe, once upon a time we were friends out of proximity, but at this point, we are doing this by choice. From France to New York, from Berlin to Detroit, and from Tunis to Chicago, I have changed, you have changed, but our friendship will always remain the same. You can change the world, I know it, and when you start believing it, know that I will be there with you. *I love you.* Taylor, your resilience, compassion, and grit inspire me. As life keeps throwing more challenges our way, we will face them with strength and grace, and be better for it. *I love you.* Rebecca, I am so glad to have met you and grown our friendship over the years. Thank you for all the help you have given me when going through pitching competitions. *I love you.*

The second group of people are the high school teachers who allowed me to get to where I am: Hélène Jousselin, Ayşin Dülger, Connie Pierce, and Christopher Conley. Mme Jousselin, il y a longtemps, vous m'avez assuré que je finirais par trouver ma place dans le monde. Aujourd'hui, j'y suis presque. Merci de m'avoir offert un instant d'existence. Mme Dülger, I have been in school for 27 years, and to this day, you remain the most extraordinary teacher I have ever had. *Thank you for caring*. Mme Pierce, I came to the U.S. for the first time at age 15 to attend Tabor, and my time there was among the hardest and loneliest of my life. You made it so much better. *Thank you*. Mr. Conley, when I first arrived at Tabor, I had no understanding of the American education system and struggled greatly to adapt; but you took the time to tell me that I was smart, that I could do better—and that is how I made it at Tabor. *Thank you*.

Last, and finally, I am so grateful for the administrators at Brown, who did their best to get us through the Kafkaesque academic bureaucracy and logistical challenges: Lauren Clarke and Jesse Polhemus. Lauren, thank you for being so kind and willing to help no matter the situation; in a world where rules break people down, you made it easier to navigate everything. Jesse, thank you for helping us record our video for our Brown venture prize. I am now working on our startup, and you were part of our origin story.

Contents

Chapter 1

Introduction

1.1 Motivation

Historically, the most successful mathematical models—those that found widespread applications across various fields—were distinguished by two key attributes: 1) their broad applicability and 2) their ability to offer comprehensive mathematical characterizations. These models—including those examined in this thesis, such as variational inequalities, pseudo-games, and Markov pseudo-games—have significantly shaped our understanding of how mathematical models can help us make sense of our reality, whether it be physics (e.g., fluid dynamics (Duvaut and Lions, 1976)), or the main object of study of this thesis: economics (e.g., asset and commodity pricing (Arrow and Debreu, 1954; Dafermos, 1990)).

While these models—most of which were developed in the early 20th century—provided mathematical insights that enabled researchers to develop a high-level understanding of the world, deriving actionable and precise conclusions required solving them. Initially, researchers would seek to obtain closed-form solutions to their models, but, as analytical solutions are intractable beyond very simple applications, with the introduction of computers in the 1960s, researchers would turn their attention to the use of algorithms to instead obtain numerical solutions for their models (Scarf, 1967a).

Thus, at the dawn of the Information Age in the 1970s, researchers, equipped with their models and building on the foundations of computational complexity theory, set their sights on discovering algorithms of broad applicability—capable of solving a vast array of mathematical optimization frameworks. In particular, computer scientists began the search for a “holy grail” algorithms with two key characteristics:

(1) broad applicability, meaning the ability to accurately solve a wide range of mathematical modeling frameworks, and (2) polynomial-time efficiency, ensuring that the algorithm halts with a solution in a number of computational steps polynomial in the problem's parameters. While some initial successes were achieved, it soon became apparent that such an algorithm might not exist at all.

Indeed, in the decades following the 1970s, beginning with the seminal work of Cook (1971) and later advanced by Papadimitriou (1994), a series of computational complexity results established that any polynomial-time algorithm capable of solving one mathematical modeling framework could, in principle, be adapted to solve another within polynomial time. While these findings established the theoretical existence of a universally applicable algorithm for all such frameworks, they also underscored a fundamental limitation: more than half a century after their introduction, no polynomial-time algorithm has been discovered for solving even a single one of these models—except in special cases.

This conjecture presents a significant challenge to the use of mathematical models in solving real-world problems. Yet, as the demand for such models grows—whether for assessing the impact of climate change or optimizing public expenditure—researchers and practitioners are often forced to rely on algorithms that terminate within a reasonable timeframe, albeit at the cost of reduced accuracy. This trend is particularly concerning, as these models are applied in high-stakes domains where inaccuracies can lead to misleading or even detrimental outcomes (see, for instance, (Kim and Kim, 2003)). *This thesis takes this challenge as its starting point, aiming to equip practitioners—particularly in economics—with an algorithmic framework for understanding and addressing the computational complexities inherent in their modeling problems.*

1.2 Scope and Thesis

Three mathematical modeling frameworks will be the object of study of this thesis:

1. **Variational inequalities** (Lions and Stampacchia, 1967): A mathematical model of problems whose set of solutions can be posed as the solutions to an inequality involving a function.

2. **Pseudo-games** (Arrow and Debreu, 1954): A mathematical model of problems whose set of solutions can be posed as the solution of a multiagent optimization problem.

3. **Stochastic pseudo-games**: A mathematical model of problems whose set of solutions can be posed as the solution of a multiagent *sequential decision-making* problem.

The two key characteristic linking these three frameworks is that a solution to them is always guaranteed to exist (under mild assumptions), and the existence of solution is established via a non-constructive fixed point theorem such as Kakutani's (Glicksberg, 1952; Kakutani, 1941) or Brouwer's fixed point theorem. In more technical terms, the problem of computing a solution to these models belongs to a class of problems known as PPAD (Papadimitriou, 1994; Daskalakis et al., 2009; Chen and Deng, 2005). While a solution to these problems is guaranteed to exist, it has now become a widely upon agreed conjecture that there does not exist an algorithm that can solve problems in the PPAD class in polynomial-time (Yannakakis, 2009).

Each of these frameworks will be used to model the problem of computing a solution to three different models of economies which belong a class of well-established economic models known as **general equilibrium models** or **infinite Walrasian economies** (i.e., highly abstract model of economies based on the demand and supply for a set of goods)[1]:

1. **(Finite) Walrasian economies**: A highly abstract model of an economy based on the demand and supply for a *finite* set of commodities.
2. **Arrow-Debreu economies**: Finite Walrasian economies in which the demand and supply is explicitly generated by consumers and firms respectively.
3. **Radner economies**: An infinite dimensional generalization of Arrow-Debreu economies which explicitly incorporates time and uncertainty in consumers' and firms' decisions.

A hierarchy of the above general equilibrium models, as well as other ones which will be discussed in the sequel, can be found in Figure 1.1.

General equilibrium models, and specifically the aforementioned models, are the foundation of much of mathematical economics, and are nowadays used in a myriad of impactful applications from resource allocation to public policy analysis. Unfortunately, the computation of a solution to these problems has also

[1] In this thesis, I use the "Walrasian economy" terminology to refer to economies with a finite set of commodities. However, Walrasian economies can in general have infinitely many commodities (Prescott and Lucas, 1972), as is the case with Radner economies which consist of a possibly infinite set of commodities, as the complete set of commodities of the economy is given by the union of commodities across the potentially infinitely many states of the economy.

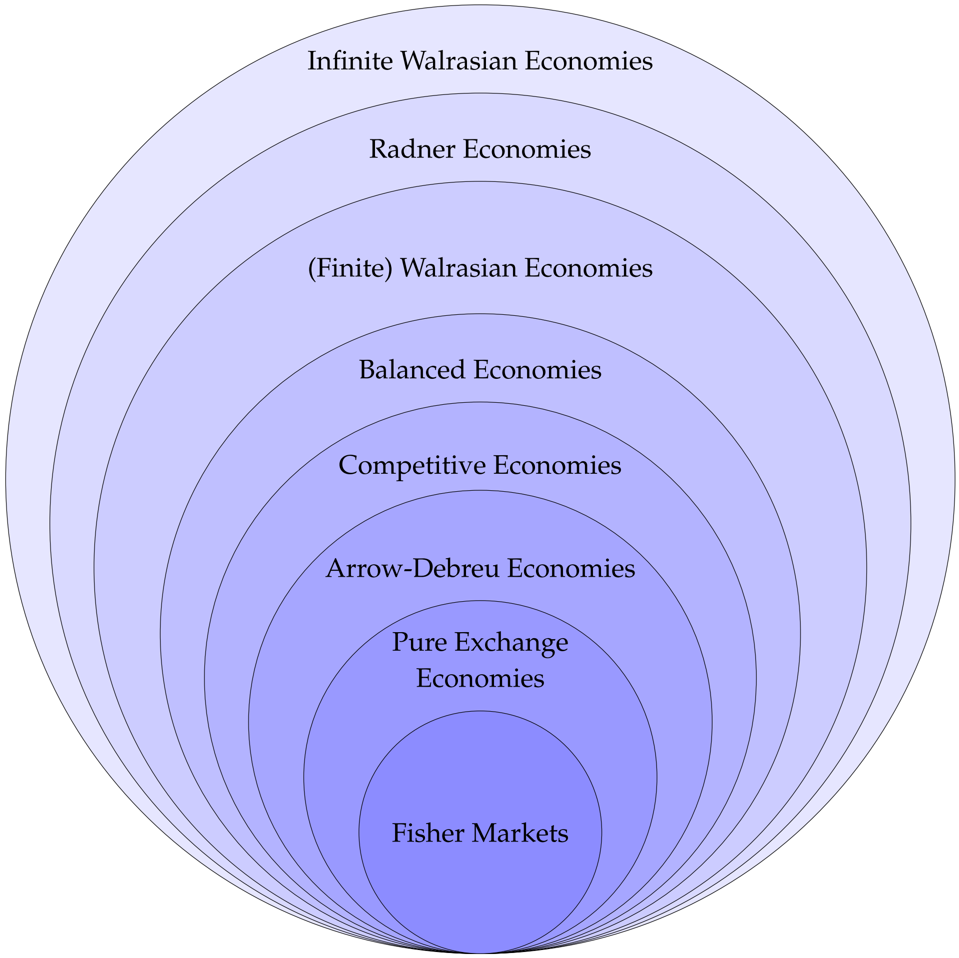


Figure 1.1: Hierarchy of the General Equilibrium Models Studied in this Thesis

been shown to belong to the class of PPAD problems, and there exists no known polynomial-time algorithm to solve them except in special cases. As such, practitioners who use these models often resort to using algorithms which halt in a reasonable amount of time at the cost of increased inaccuracy. This practice raises concerns, as these algorithms are for instance used to solve models for public policy at central banks, with inaccurate solutions often leading to disastrous policy recommendations (see, for instance Kim and Kim (2003)).

In light of these discouraging facts, I will take an optimistic stance, and defend the following thesis.

Thesis

> *There exists a meaningful algorithmic theory of general equilibrium which can allow practitioners to effectively trade-off accuracy and computational efficiency, and provides broadly-applicable algorithms which perform well in practice.*

To argue this position, I will seek to answer the following twin questions:

1. Can we develop a unified mathematical and computational framework for solving general equilibrium models in a systematic way?
2. Can we develop broadly applicable algorithms which perform well in practice, and whose performance can be explained by this framework despite existing impossibility results?

My answer will be two-folds:

1. I will provide characterizations of Walrasian, Arrow-Debreu, and Radner economies using the VI, pseudo-game, and Markov pseudo-game modeling frameworks respectively, and apply the computational tools for these framework to approach the computation of a solution to these general equilibrium models.
2. I will devise broadly applicable methods that accurately and efficiently solve these general equilibrium models in practice, and provide a theoretical explanation of their high performance using the computational properties of the aforementioned mathematical modeling frameworks.

1.3 Outline

This thesis is organized in 3 major parts which are presented following a review of the necessary mathematical background to understand its content in Chapter 2. For readers familiar with the mathematical background, I include a "*too long; did not read*" section (Section 2.1) which summarizes the mathematical notation adopted in this thesis.

Each part of this thesis is broken down into two main chapters with the first chapter of each part describing an optimization framework, its mathematical and algorithmic properties, and the second chapter describing its application to a type of general equilibrium model.[2] I summarize the outline of these three major parts below.

OUTLINE OF MAJOR THESIS PARTS

(Part I) Variational Inequalities and Walrasian Economies

- (Chapter 4) Variational Inequalities
- (Chapter 5) Walrasian Economies

(Part II) Pseudo-Games and Arrow-Debreu Economies

- (Chapter 9) Pseudo-Games
- (Chapter 10) Arrow-Debreu Economies

(Part III) Markov Pseudo-Games and Radner Economies

- (Chapter 12) Markov Pseudo-Games
- (Chapter 13) Radner Economies

Each chapter which concerns optimization frameworks consists of three major sections: 1) mathematical background, 2) first-order methods to solve the optimization framework, 3) merit function methods to solve the optimization framework. The definitions of "first-order method" and "merit function method" is described precisely in the chapter relevant to each optimization framework.

Similarly, each chapter which concerns general equilibrium models consists of three major sections: 1) mathematical background and equivalence with the optimization framework presented in the preceding chapter, 2) application of the first-order method presented in the preceding chapter, 3) application of the merit-function method presented in the preceding chapter

The organization of the parts in the above given order has three purposes. First, the results introduced in Chapter 4 are used in the rest of the thesis thus requiring Part I to come before all others. Second, VIs (resp. Walrasian economies) can model as special cases pseudo-games (resp. Arrow-Debreu economies) thus the

[2]Part I, in addition to these two main chapters also includes an additional chapter on Fisher markets (Chapter 6) as a specific example of Walrasian economies for illustrative purposes.

results in Part I can provide additional insights on the results in Part II. Third, Markov pseudo-games (resp. Radner economies) can be seen as infinite dimensional (resp. infinitely many commodity) generalizations of VIs and pseudo-games (resp. Walrasian economies and Arrow-Debreu) economies, which the literature has started exploring only in more recent years, and as such the results in Part I and Part II allow to contextualize the results in Part III more effectively, and provides an open-ending for a new and exciting research direction on infinite-dimensional optimization and economies with infinitely many commodities.

1.4 Contributions

A high-level summary of the contributions of each chapter of the thesis can be found below. For a more detailed explanation of the contributions of each chapter, I refer the reader to the Contributions Section of the part in which the chapter is contained (Sections 3.3, 8.3, 11.3). Additionally, a summary of the main computational results for optimization frameworks and general equilibrium models can be found in Table 1.2 and Table 1.1 respectively.

(CHAPTER 4: VARIATIONAL INEQUALITIES): The mirror extragradient class of algorithms (Algorithm 3, Chapter 4) is introduced, with best-iterate polynomial-time convergence established for variational inequalities (VIs) that satisfy the Minty condition and are pathwise Bregman-continuous (e.g., Lipschitz-continuous VIs)[3]. This result generalizes the extragradient method analysis of Huang and Zhang (2023) and extends the convergence guarantees of Zhang and Dai (2023) from unconstrained to constrained domains. Additionally, conditions for the *local* convergence of the mirror extragradient algorithm to an ε-strong solution in Bregman-continuous VIs *in the absence of* the Minty condition are established, representing the first known result of this kind. Finally, for general VIs, a polynomial-time glob-

[3]Bregman-continuity is a generalization of Lipschitz-continuity in terms of the Bregman divergence, while pathwise Bregman-continuity is a further weakening of the Bregman-continuity condition, which requires Bregman-continuity to only hold over trajectories of the algorithm. See, Chapter 4 for additional definitions and explanation.

ally convergent class of merit function methods is developed to compute a solution satisfying the necessary conditions for a Walrasian equilibrium.

(CHAPTER 5: WALRASIAN ECONOMIES): A computationally tractable characterization of Walrasian equilibria in balanced economies is established as the strong solutions of a variational inequality (VI) satisfying the Minty condition within a unit box constraint. This leads to the introduction of the mirror *extratâtonnement process* (Algorithm 6, Chapter 5), a novel price-adjustment method based on the mirror extragradient approach, whose polynomial-time convergence is proven in all balanced economies satisfying pathwise Bregman-continuity. Polynomial-time convergence is further demonstrated in competitive economies that are variationally stable under bounded excess demand elasticity, extending prior polynomial-time *tâtonnement* convergence results under the Gross Substitutes (GS), Weak Gross Substitutes (WGS) and Weak Axiom of Revealed Preferences (WARP) conditions (see, Figure 1.2 for additional details, and Chapter 5 for precise definitions) to a much larger class. The process is also shown to converge in polynomial time within the Scarf economy, marking the first such result for a natural discrete-time price-adjustment method. Experimental validation confirms the theoretical assumptions and demonstrates efficient computation of Walrasian equilibria in large-scale competitive economies, including PPAD-complete cases such as Leontief economies, achieving fast and reliable convergence. Finally, for general, potentially non-balanced Walrasian economies, a polynomial-time globally convergent class of merit function methods is developed to compute a solution satisfying the necessary conditions for a Walrasian equilibrium.

(CHAPTER 6: HOMOTHETIC FISHER MARKET): The maximum absolute value of the Hicksian price elasticity of demand is identified as a key parameter for analyzing the convergence of (entropic) tâtonnement in an important class of Walrasian economies known as homothetic Fisher markets. A sublinear convergence rate of $O(^{(1+\epsilon^2)}/_T)$ is established, where ϵ represents the maximum Hicksian price elasticity across buyers. This result generalizes existing convergence analyses for CES and nested CES utilities, unifying previously disjointed convergence and non-convergence findings. It encompasses the full spectrum of (nested) CES utilities, including Leontief and linear utilities, recovering the best-known rate of $O(^1/_T)$ for Leontief markets ($\epsilon = 0$) and confirming the non-convergent behavior of tâtonnement in linear markets as $\epsilon \to \infty$. Known existing computation results for the convergence of tâtonnement in Fisher markets in light of this result is summarized in Figure 3.1a.

(CHAPTER 9: PSEUDO-GAMES) The existence of variational equilibrium in quasiconcave pseudo-games with jointly convex constraints is first reestablished, followed by the introduction of first-order variational equilibrium, which is shown to exist in a broader class of pseudo-games—specifically, smooth games with jointly convex constraints. An equivalence is then established between (first-order) variational equilibria of pseudo-games and strong solutions of variational inequalities, leading to the characterization of a new class of pseudo-games, termed variationally stable pseudo-games with jointly convex constraints. For this class, first-order variational equilibrium can be computed in polynomial time using a novel learning dynamic called the mirror extragradient learning dynamics (Algorithm 7, Chapter 9). In the special case where the pseudo-game is also concave, this result extends to variational equilibrium computation, representing the broadest known result of its kind. Finally, for general pseudo-games with jointly convex constraints that are not necessarily variationally stable, a polynomial-time globally convergent class of merit function methods is developed to compute a solution that satisfies the necessary conditions for variational equilibrium.

(CHAPTER 10: ARROW-DEBREU ECONOMIES): Novel mathematical characterizations of Arrow-Debreu equilibrium in Arrow-Debreu economies are developed. First, it is re-established that the set of Arrow-Debreu equilibria in any quasiconcave Arrow-Debreu economy corresponds to the set of generalized Nash equilibria of the Arrow-Debreu pseudo-game. Due to the intractability of this characterization, a new formulation is introduced, expressing the set of Arrow-Debreu equilibria in any concave pure exchange economy as the set of generalized Nash equilibria of the trading post pseudo-game, a variationally stable pseudo-game with jointly convex constraints. The mirror extragradient learning dynamics are then applied to this pseudo-game, yielding a market dynamic termed the mirror extratrade dynamics. While the trading post pseudo-game is not concave, it is shown to be pseudoconcave, allowing an approximate first-order variational equilibrium to be computed in polynomial time, with asymptotic convergence to a variational equilibrium—and thus to an Arrow-Debreu equilibrium of the associated concave pure exchange economy—to the best of my knowledge the broadest convergence result of its kind. Finally, for general, potentially non-concave Arrow-Debreu economies, a polynomial-time globally convergent class of merit function methods is developed to compute a solution satisfying the necessary conditions for an Arrow-Debreu equilibrium.

(CHAPTER 12: MARKOV PSEUDO-GAMES): Markov pseudo-games are introduced as a generalization of Markov games, where other players' actions influence both rewards and available actions. The existence of pure generalized Markov perfect equilibria (GMPE) is established in concave Markov pseudo-games, extending to the stochastic setting, Arrow-Debreu's equilibrium existence results for concave pseudo-games. This also implies the existence of pure Markov perfect equilibria in a broad class of continuous-action Markov games, where previously only mixed-strategy equilibria were known (Fink, 1964; Takahashi, 1964). Although computing GMPE is PPAD-hard in general, the problem is reformulated as a generative adversarial learning task, where a generator proposes an equilibrium policy profile, and an adversary produces best responses. Leveraging advances in generative adversarial learning, it is shown that under mild assumptions, a policy profile that is a stationary point of exploitability (players' cumulative maximum regret) can be computed in polynomial time. This result applies to Markov pseudo-games with a bounded best-response mismatch coefficient, ensuring that states explored by any GMPE are sufficiently sampled under the initial state distribution. The approach parallels known computational results for zero-sum Markov games. As these theoretical guarantees hold for policies represented by neural networks, this provides the first deep reinforcement learning algorithm with theoretical guarantees for general-sum games.

(CHAPTER 13: RADNER ECONOMIES): An extension of Magill and Quinzii's infinite horizon exchange economy (Magill and Quinzii, 1994), termed the Radner economy, is introduced, generalizing the model to arbitrary assets while restricting the transition dynamics to be Markovian. This restriction enables the proof of existence for a recursive Radner equilibrium (RRE), a Radner equilibrium independent of the initial state distribution, simplifying the equilibrium policy domain to the space of states rather than histories and making computation more tractable. The set of RREs in any Radner economy is reformulated as the set of generalized Markov perfect equilibria (GMPE) of an associated Markov pseudo-game, extending prior results that were limited to economies with a single consumer, commodity, or asset. This formulation further implies that a stationary point of exploitability in the associated Markov pseudo-game can be computed in polynomial time. To validate these theoretical results, a the method is implemented as a generative adversarial policy network and applied to three Radner economies with distinct utility functions. Experimental findings indicate that the method produces approximate equilibrium policies that are much closer to GMPE than those generated by a standard baseline for solving stochastic economies.

Part	Chapter	Economy Class	Economy Subclass	Solution Concept	Algorithm	Convergence Rate	Result Reference
I	5	Walrasian Economies	Balanced + Pathwise Bregman-Continuous	Walrasian Equilibrium	Mirror Extratâtonnement	$O(1/\sqrt{\tau})$	Theorem 5.4.1
I	5	Walrasian Economies	Competitive + Variationally Stable + Elastic + Bounded	Walrasian Equilibrium	Mirror Extratâtonnement	$O(1/\sqrt{\tau})$	Theorem 5.4.2
I	5	Walrasian Economies	Scarf Economy	Walrasian Equilibrium	Mirror Extratâtonnement	$O(1/\sqrt{\tau})$	Corollary 5.4.4
I	5	Walrasian Economies	Lipschitz-Continuous + Lipschitz-Smooth	Stationary Point of Walrasian Merit Function	Mirror Potential	$O(1/\tau)$	Theorem 5.5.1
I	6	Walrasian Economies	Homothetic Fisher Markets + Bounded Hicksian Demand Elasticity	Walrasian Equilibrium	Entropic Tâtonnement	$O(1/\tau^2)$	Theorem 6.5.1
II	10	Arrow-Debreu Economies	Concave Pure Exchange Economy + Lipschitz-Smooth Utilities	Arrow-Debreu Equilibrium	Mirror Extratrade Dynamics	$O(1/\sqrt{\tau})$	Theorem 10.3.1
II	10	Arrow-Debreu Economies	Quasiconcave + Lipschitz-Smooth Utilities	Stationary Point of Arrow-Debreu Exploitability	REDA	$O(1/\tau)$	Theorem 10.4.1
II	10	Arrow-Debreu Economies	Lipschitz-Smooth Utilities + Lipschitz-Smooth Utility Gradients	Stationary Point of Arrow-Debreu Variational Exploitability	Mirror Variational Dynamics	$O(1/\tau)$	Theorem 10.4.2
III	13	Radner Economies	Concave + Lipschitz-Smooth Utilities	Stationary Point of Radner (State) Exploitability	TTSGDA	$O(1/\sqrt[6]{\tau})$	Theorem 10.4.1

Table 1.1: Summary of Main Computational Results for General Equilibrium Models

Part	Chapter	Optimization Framework	Class of Framework	Solution Concept	Algorithm	Convergence Rate	Result Reference
I	4	Variational Inequalities	Minty + Pathwise Bregman-Continuous	Strong (Stampacchia) Solution	Mirror Extragradient Algorithm	$O(1/\sqrt{\tau})$	Theorem 4.3.1
I	4	Variational Inequalities	"Local" Minty + Bregman-Continuous	Strong (Stampacchia) Solution	Mirror Extragradient Algorithm	$O(1/\sqrt{\tau})$ Local Convergence	Theorem 4.3.2
I	4	Variational Inequalities	Lipschitz-Continuous + Lipschitz-Smooth	Stationary Point of Regularized Primal Gap	Mirror Potential Algorithm	$O(1/\tau)$	Theorem 4.4.1
II	9	Pseudo-Games	Concave + Variationally Stable + Lipschitz-Smooth + Jointly Convex Constraints	Variational Equilibrium	Mirror Extragradient Dynamics	$O(1/\sqrt{\tau})$	Theorem 9.4.1
II	9	Pseudo-Games	Variationally Stable + Lipschitz-Smooth + Jointly Convex Constraints	First-Order Variational Equilibrium	Mirror Extragradient Dynamics	$O(1/\sqrt{\tau})$	Theorem 9.6.1
II	9	Pseudo-Games	Concave + Lipschitz-Smooth + Jointly Convex Constraints	Stationary Point of Regularized Exploitability	REDA	$O(1/\tau)$	Theorem 9.4.3
II	9	Pseudo-Games	Lipschitz-Smooth Payoff + Lipschitz-Smooth Payoff Gradients + Jointly Convex Constraints	Stationary Point of Variational Exploitability	Mirror Variational Dynamics	$O(1/\tau)$	Theorem 9.6.2
III	12	Markov Pseudo-Games	Concave + Lipschitz-Smooth	Stationary Point of (State) Exploitability	TTSGDA	$O(1/\sqrt[6]{\tau})$	Theorem 12.3.1

Table 1.2: Summary of Main Computational Results for Optimization Frameworks

1.5 Historical and Academic Context

1.5.1 General Equilibrium Theory: The Foundations of Economic Modeling

Key to the historical development of mathematical models in economics was a need to understand how economies functioned (i.e., the emergence of demand, supply, and prices) dating back to as early as the 18th century to French-Irish economist Richard Cantillon's work (Cantillon, 1755). While a great number of economists including Adam Smith (Smith, 1937), David Ricardo (Ricardo, 1895), John Stuart Mill (Mill, 1965), and Alfred Marshall (Marshall, 1910) would make great contributions to our understanding of demand,

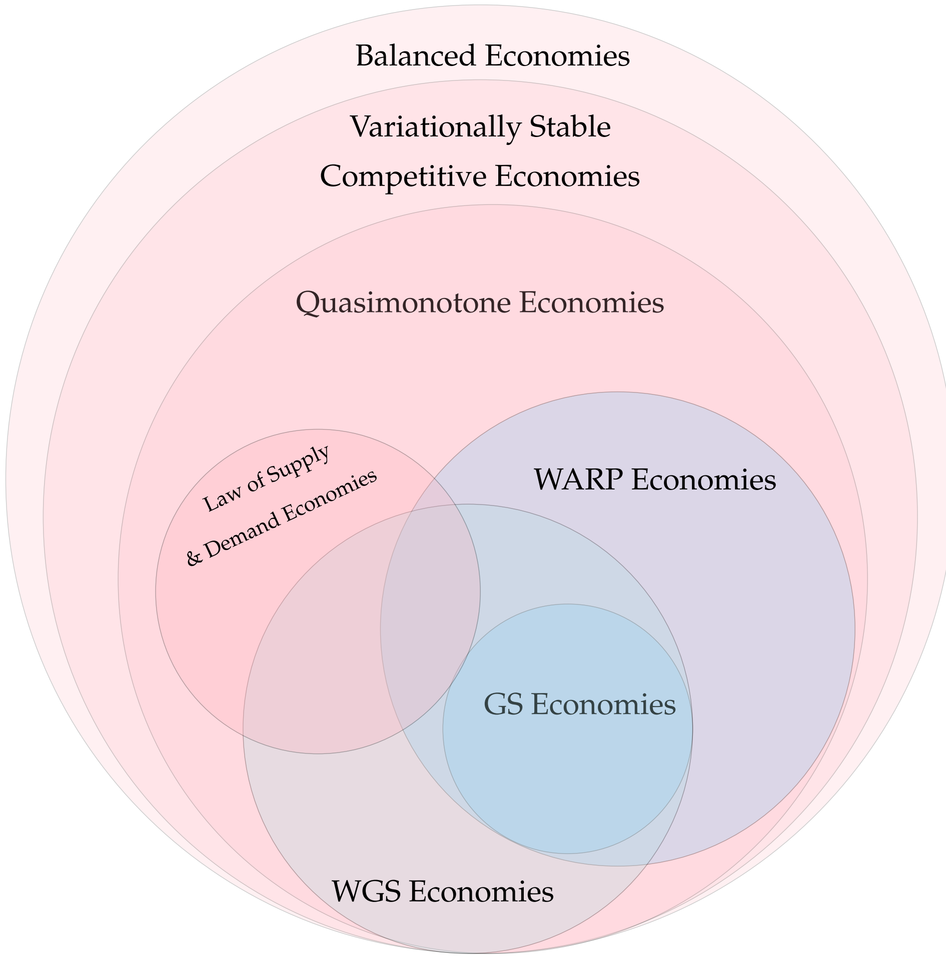
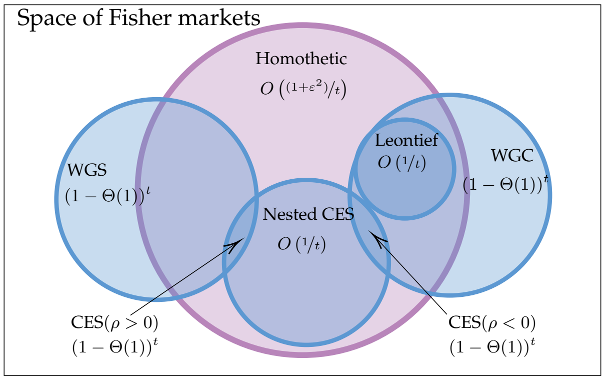


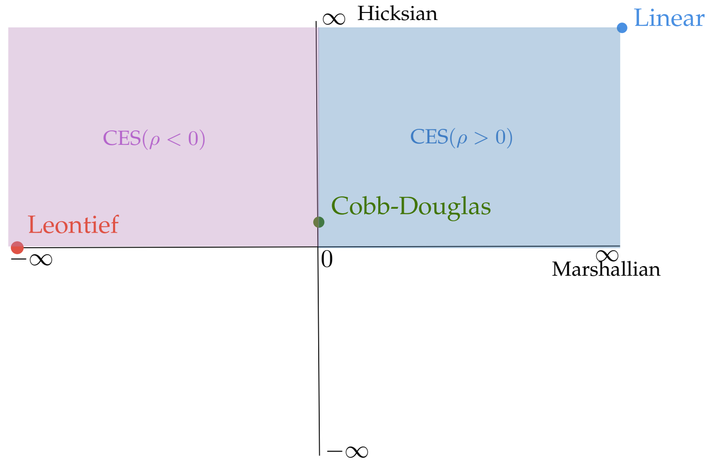
Figure 1.2: Walrasian Economies for which there exists a polynomial-time price-adjustment process. Economies for which I prove polynomial-time convergence of mirror *extratâtonnement* are depicted in pink hues (i.e., balanced, variationally stable competitive, quasimonotone, law of supply and demand economies), while economies for which polynomial-time convergence was known are depicted in blue hues (i.e., WGS, GS, WARP economies). The convergence result for balanced economies holds under the assumption of pathwise Bregman-continuity whose plausibility I verify in experiments (Section 5.4.3, Chapter 5, Part I), while the convergence result for variational stable competitive economies hold under the assumption that the price elasticity of excess demand is bounded (see, Chapter 5, Part I) for additional detail. The polynomial-time price-adjustment process for economies in blue is *tâtonnement*. The convergence result for WARP economies was introduced by Uzawa (1960), the first weakly polynomial-time convergence result (i.e., without any elasticity boundedness assumptions) for WGS economies was introduced by Codenotti et al. (2005), with Cole and Fleischer (2008) proving a strongly polynomial-time convergence result (i.e., in terms of elasticity bounds).

supply, and prices, it would not be until the pioneering work of French economist Léon Walras (Walras, 1896) that a clear modeling paradigm for economies would start appearing.

Walras formulated a mathematical model of markets (nowadays known as a **Walrasian market**) as a system of resource allocation comprising supply and demand functions that map values for resources, called **prices**, to quantities of resources—*ceteris paribus*, i.e., all else being equal. Walras also defined a steady state of a



(a) The convergence rates of *tâtonnement* for different Fisher markets. We color previous contributions in blue, and our contribution in red, i.e., we study homothetic Fisher markets where ϵ is the maximum absolute value of the price elasticity of Hicksian demand across all buyers. We note that the convergence rate for WGS markets does not apply to markets where the price elasticity of Marshallian demand is unbounded, e.g., linear Fisher markets; likewise, the convergence rate for nested CES Fisher markets does not apply to linear or Leontief Fisher markets.



(b) Cross-price elasticity taxonomy of well-known homogeneous utility functions. There are no previously studied utility functions in the space of utility functions with negative Hicksian cross-price elasticity. Future work could investigate this space and prove faster convergence rates than those provided in this paper. We note that our convergence result covers the entire spectrum of this taxonomy (excluding limits of the y-axis).

Figure 1.3: A summary of known results in Fisher markets.

market, which he called **competitive** (nowadays, also called a **Walrasian**) **equilibrium**, as prices s.t. the demand is **feasible**, i.e., the demand for each resource is less than or equal to its supply, and **Walras' law** holds, i.e., the value of the supply is equal to the value of the demand. Unlike in Walras' model, all else is not equal, and real-world markets do not exist in isolation but are part of an **economy**. Indeed, the supply and demand of resources in one market depend not only on prices in that market, but also on the supply and demand of resources in other markets. If every market in an economy is simultaneously at a competitive equilibrium, Walras' law holds for the economy as a whole; this steady state, now a property of the economy, is called a **general equilibrium**.

As such, Walras' early foray in economic modeling would leave two important questions open. First, it would be unclear if Walras' market model could model an entire economy with consumers and firms and a number of different markets. Second, Walras did not provide conditions that guarantee the existence of a competitive equilibrium and it was not clear if such prices existed. The question of whether Walras' model could be seen as a model of an economy, and under what conditions competitive equilibrium prices existed, would remain open for over half a century until Arrow and Debreu's seminal **model of a competitive**

economy (nowadays also called **Arrow-Debreu economies**) for which Arrow and Debreu (1954) proved the existence of a competitive equilibrium.

Remark 1.5.1 [Some history]**.**
Following the second world war, European and American governments had to preside over the reconstruction of devastated economies. This required an improved understanding of the role of public expenditure in economic activity since the war led to an unprecedented increase in the role of government in the economy, a change that was unsustainable: While in 1944, the American government's spending at all levels accounted for 55 percent of gross domestic product (GDP), by 1947, government spending had dropped 75 percent in real terms, or from 55 percent of GDP to just over 16 percent of GDP (Bureau of Economic Analysis, 2021). Fortunately, when the Second World War (WW2) broke out, many European academics escaping the war moved to the United States helping the development of rigorous mathematical models of economies which would be key to help manage the role of public expenditure in economic growth. These academics re-emerged primarily at the University of Chicago, where the Cowles Commission was founded in 19391, with the maxim, "Science is Measurement" (Mitra-Kahn, 2005). The Cowles commission aimed to link mathematics and economics (Mitra-Kahn, 2005), and played a crucial role in the development of mathematical microeconomic models that have become the foundation of modern economics. These efforts that were initiated by the Cowles Commision culminated to the seminal work of Keneth Arrow and Gérard Debreu which proved the existence of general equilibria in a very general setting (Arrow and Debreu, 1954).

In their model, Arrow and Debreu posit a set of resources, modeled as commodities, each of which is assigned a price; a set of consumers, each choosing a quantity of each commodity to consume that maximizes their utility function in exchange for their endowment (e.g., labor); and a set of firms, each choosing a profit-maximizing quantity of each commodity to produce, with prices determining **aggregate demand**, i.e., the sum of the utility-maximizing consumptions across all consumers, and **aggregate supply**, i.e., the sum of endowments and profit-maximizing productions across all consumers and firms, respectively. Notably, Arrow and Debreu's model could be seen as a special case of Walras model defined by the aggregate demand and aggregate supply, and its solution being defined as the associated competitive, now better called general, equilibrium. This reduction would demonstrate that Walras' much simpler model can accurately model an economy without the need to assume that all else is equal, allowing us to call his model a **Walrasian economy** rather than a market *de jure*. Even more importantly, under mild assumptions Arrow-Debreu proved that a general equilibrium in their model existed, hence providing sufficient and meaningful conditions for the existence of a competitive equilibrium in Walrasian economies.

Nevertheless, soon after Arrow and Debreu's monumental achievement, a third issue would emerge. Arrow-Debreu and Walrasian economies are static economies which do not explicitly model time and uncertainty. That is, unlike real-world economies, all trade in such economies happens in one single time period with the world being at a given state. To get around this issue, Arrow and Debreu argued that commodities were to be seen as state and time contingent, with each one representing a good or service which can be bought or sold in a single time period, but that encodes delivery opportunities at a finite number of distinct points in state and time. However, as this explanation would not be a realistic explanation of real-world economies, in the decades to come economists would seek to answer whether if Arrow-Debreu and Walrasian economies could represent an economy with time and uncertainty, and if a general equilibrium would be guaranteed to exist in economies with time and uncertainty.

The question would mostly remain open for nearly 20 years until Radner's introduction of the **stochastic competitive economy** (nowadays also called the **Radner economy** in which he proved the existence of a general equilibrium under suitable assumptions. The Radner economy, initialized at a state of the world, is a finite-horizon economy comprising a sequence of **spot markets** in which consumers and firms can purchase and sell commodities for immediate delivery, all linked across time by **asset markets** in which consumers and firms can buy or sell assets that deliver a payment when particular state of the world occurs, with the economy stochastically transitioning to one of many other world states once consumers and firms have made their purchase. Commodities (resp. assets) are assigned state and time contingent prices which determine their **aggregate state and time contingent demand**, i.e., the sum of utility maximizing consumptions (resp. asset portfolios), and **aggregate state and time contingent supply**, i.e., the sum of endowments and profit-maximizing productions (resp. asset portfolios) across all consumers and firms, respectively. Similar to Arrow-Debreu economies, any Radner economy can be cast as a Walrasian economy given by the aggregate state and time contingent demand and supply, with its solution being defined as a general equilibrium of this Walrasian economy. Further, under suitable assumptions on the asset market Arrow (1964) shows that any Radner economy can also be represented as an Arrow-Debreu economy, thus further demonstrating that Walrasian economies and their solution concept the general equilibrium can effectively model economies with time and uncertainty.

Nearly half a century after their introductions Arrow-Debreu and Radner economies have become foundational pillars of modern mathematical economics, providing an explanation of the most important facets of any economy. While these models only scratch the surface of the mathematical models of economies developed since Arrow and Debreu's work, all of the developed models share one common characteristic: they can be cast as Walrasian economies with their solution corresponding to a general equilibrium of the associated Walrasian economy, thus leading these models to be colloquially called **general equilibrium models**.

In developing their models and establishing their existence results Arrow and Debreu and Radner would pioneer the development of theory of games and stochastic games which would play a key role in the development of various other branches of mathematical economics such as mechanism design and financial economics. While the algorithmic theory developed in this thesis will be applied specifically to the analysis of algorithms for general equilibrium models, all the results developed within this thesis will be first developed within broader and more abstract mathematical optimization frameworks some of which first studied by Arrow and Debreu (e.g., pseudo-games) and Radner (e.g., stochastic pseudo-games), and others developed subsequently (e.g., variational inequalities). As such, the algorithms and analyses I provide in this thesis is relevant not only to general equilibrium models but also to other areas of mathematical economics such as mechanism design.

1.5.2 General Equilibrium Theory at the Origin of Mechanism Design

Much of mathematical microeconomic theory following the 1970s would focus its resources on the development of mechanism design (i.e., mathematical and algorithmic frameworks for the design of markets) which has nowadays become a cornerstone of mathematical economics. In particular, the computational literature on economics has dedicated a great deal of resources often at the expense of general equilibrium theory, to the development of an algorithmic theory of mechanism design, perhaps due to the financial incentives provided by the emergence of online market places. The goal of this digression is to clarify the historical connections between mechanism design, and underline the importance of developing the algorithmic theory of general equilibrium to further our understanding of algorithmic mechanism design.

Recall that the key political driver behind the development of general equilibrium theory was the need to better understand economies in order to optimally reduce public expenditure, which had drastically increased during World War II. (see Remark 1.5.1 for additional details). As part of this development, in separate papers, Arrow and Debreu independently but simultaneously proved the first and second welfare theorems of economics, which stated that 1) consumptions and productions associated with a general equilibrium are Pareto-efficient and 2) any collection of Pareto-optimal consumptions and productions can be associated with a general equilibrium (price system) (Arrow, 1951a; Debreu, 1951b).

Arrow and Debreu's results implied that competitive economies[4] are then an efficient way to allocate resources since they result in a Pareto-optimal distribution of resources—an inference which is contingent on the global stability of general equilibria, i.e., free markets actually settling into a general equilibrium. Unfortunately, Arrow and Debreu's results provided ambiguous conclusions on how to transition away from a war economy with high public expenditure. On one hand, Arrow and Debreu's results suggested that a post-war economy with no governmental intervention and no public expenditure was optimal as free markets are a Pareto-efficient mechanism of resource allocation. On the other hand, real-world markets rarely would ever truly be free and it seemed like decreasing public expenditure significantly could be a disastrous economic choice as Paul Samuelson, 1970 Nobel Prize laureate, wrote in 1943: "some ten million men will be thrown on the labor market" (Mitra-Kahn, 2005), warning that it would be "the greatest period of unemployment and industrial dislocation which any economy has ever faced" (Mitra-Kahn, 2005).

The issue of public expenditure in a post-war world would become a central theme in Samuelson's research who would provide the first rigorous definition of public goods. Based on his definition Samuelson would then derive what came to be known as the Samuelson condition: the first order optimality condition associated with the optimal provision of public goods in terms of the demand and supply for public goods

[4]A competitive economy in this sense is one that resembles an Arrow-Debreu economy.

(Samuelson, 1954). Samuelson's work was in some sense groundbreaking as it moved the problem of allocating public spending from the realm of political theory to the realm of general equilibrium theory. Samuelson's analysis would be studied more rigorously in subsequent general equilibrium models culminating eventually in a generalization of Arrow-Debreu's model of a competitive economy called the private ownership economy with public goods, a model which differentiates between public and private goods (Mitra-Kahn, 2005).

An important result that emerged from this line of work started by Samuelson and which was proven by Foley is that a general equilibrium (also known as a **Lindahl-Foley equilibrium** exists and the first and second welfare theorems apply in a general equilibrium model with private ownership and public goods (also known as the **Lindahl-Foley economy**) (Foley, 1970). Foley's results confirmed the role that governments have been attributed in achieving a Pareto-optimal allocation of resources via redistribution policies (Mitra-Kahn, 2005). Although a gross simplification of the conclusion, this meant that governments could direct their public expenditures using the first-order optimality conditions determined by Samuelson, which now could be interpreted as the welfare maximizing quantities of public goods prescribed by the general equilibrium of the Lindahl-Foley economy, to provide optimal levels of public goods, ensuring their economies settle into a Pareto-optimal allocation of resources!

Unfortunately, while a Lindahl-Foley equilibrium provides us with a way to determine the optimal quantities of public goods to provide, it turns out that the computation of the optimal provision of public goods is not straightforward since it requires the policy maker to know the true preferences of consumers in the Lindahl-Foley economy. One possible workaround to this issue is to elicit the preferences of consumers; however, as it turns out, consumers have an incentive to lie about their preferences over public goods, as they can obtain a more favorable allocation of public goods prescribed by the Lindahl-Foley equilibrium by doing so. This in turn might lead the policy maker to compute a non-optimal provision of public goods.

The issue of incentive-compatibility, i.e., the consumers not reporting their preferences truthfully, in the provision of public goods is exactly what led to the development of mechanism design. In order to ensure that economies achieve a Pareto-optimal allocation of resources, governments had to decide their equilibrium level of public expenditure, yet to compute their equilibrium level expenditures, governments had to be able to elicit the true preference of consumers over public goods. This required the development of the mechanism design literature which would provide a new formalization of social and economic interactions which accounted for incentives. The consistent objective of the mechanism design literature which was introduced in seminal papers by Hurwicz in the 60s and 70s (Hurwicz, 1972; 1960; 1979), was in fact to provide a more powerful framework than general equilibrium theory to address the issue of incentive-compatibility. This means that mechanism design should be seen as a generalization of general equilibrium theory. More precisely, the general equilibrium or the competitive equilibrium is a particular outcome of a game just like the VCG outcome is, and the Walrasian mechanism, i.e., the function which takes as input the preferences of consumers, computes and outputs a competitive equilibrium, is an instance of a mechanism just like the VCG mechanism is.

More importantly, however, history tells us that without general equilibrium theory, there would be no mechanism design. Perhaps an analogy is fitting here: General equilibrium is to mechanism design, as the normal distribution is to probability theory. Just like one cannot imagine a theory of probability without a solid understanding of the statistical and algorithmic properties of the normal distribution, one cannot expect a proper understanding of mechanism design without a proper understanding of the economic and algorithmic properties of general equilibria. History is the biggest testament to this statement; without general equilibrium theory's incentive issue, Hurwicz and many others would not have gotten inspired to introduce mechanism design. As such, the algorithms and analyses introduced in this thesis inform not only general equilibrium theory but also other branches of mathematical economics such as mechanism design.

Chapter 2

Mathematical Background

2.1 Too Long; Did Not Read

For the reader familiar with the mathematical background, in this section, I provide a brief overview of the notational conventions and mathematical definitions. For the remainder of this thesis, excluding the conclusion, I will use the pronoun "we" instead of "I" to maintain narrative fluidity in the mathematical exposition.

2.1.1 Notation

In general, we allow for the use of lowercase letters to denote any mathematical object, and adopt the following calligraphic conventions to insist on the nature of the mathematical object at hand: We use caligraphic uppercase letters to denote sets (e.g., $\mathcal{X}$), bold uppercase letters to denote matrices (e.g., $\boldsymbol{X}$), bold lowercase letters to denote vectors (e.g., $\boldsymbol{p}$), lowercase letters to denote scalar quantities (e.g., x), and uppercase letters to denote random variables (e.g., X). We denote the ith row vector of a matrix (e.g., $\boldsymbol{X}$) by the corresponding bold lowercase letter with subscript i (e.g., $\boldsymbol{x}_i$). Similarly, we denote the jth entry of a vector (e.g., $\boldsymbol{p}$ or $\boldsymbol{x}_i$) by the corresponding lowercase letter with subscript j (e.g., p_j or x_{ij}). We denote functions by a letter determined by the value of the function, e.g., f if the mapping is scalar valued, $\boldsymbol{f}$ if the mapping is vector valued, and $\mathcal{F}$ if the mapping is set valued.

For any collection of sets $\{\mathcal{A}_i\}_{i\in[n]}$, we define the notation $(\boldsymbol{a}_i, \boldsymbol{a}_{-i}) \doteq (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) \in \bigtimes_{i\in[n]} \mathcal{A}_i$, where $\boldsymbol{a}_{-i} \in \bigtimes_{i'\in[n], i'\neq i} \mathcal{A}_{i'}$ denotes $(\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n)$ with the ith entry $\boldsymbol{a}_i \in \mathcal{A}_i$ removed.

We denote the set $\{1, \ldots, n\}$ by $[n]$, the set $\{n, n+1, \ldots, m\}$ by $[n:m]$, the set of natural numbers by $\mathbb{N}$, and the set of real numbers by $\mathbb{R}$. We denote the positive and strictly positive elements of a set using a $+$ or $++$ subscript, respectively, e.g., $\mathbb{R}_+$ and $\mathbb{R}_{++}$. For any $n \in \mathbb{N}$, we denote the n-dimensional vector of zeros and ones by $\mathbf{0}_n$ and $\mathbf{1}_n$, respectively.

2.1.2 Mathematical Definitions

We let $\Delta_n = \{\boldsymbol{x} \in \mathbb{R}_+^n \mid \sum_{i=1}^n x_i = 1\}$ denote the unit simplex in $\mathbb{R}^n$, and $\Delta(A)$ denote the set of all probability measures over a given set A. We also define the support of a probability density function $f \in \Delta(\mathcal{X})$ as $\mathrm{supp}(f) \doteq \{\boldsymbol{x} \in \mathcal{X} : f(\boldsymbol{x}) > 0\}$. Finally, we denote the orthogonal projection operator onto a set C by Π_C, i.e., $\Pi_C(\boldsymbol{x}) \doteq \arg\min_{\boldsymbol{y} \in C} \|\boldsymbol{x} - \boldsymbol{y}\|^2$.

For any $\varepsilon \geq 0$, we write $\mathcal{B}_\varepsilon[x] = \{x' \in \mathcal{M} \mid d(x, x') \leq \varepsilon\}$ and $\mathcal{B}_\varepsilon(x) = \{x' \in \mathcal{M} \mid d(x, x') \leq \varepsilon\}$ to respectively denote the closed and open ε-ball centered at $x \in \mathcal{M}$.

For any real number $a \in \mathbb{R}$, $a\mathcal{X}$ denotes the (Minkowski) product, i.e., $a\mathcal{Y} \doteq \{ax \mid x \in \mathcal{X}\}$. $\mathcal{X} + \mathcal{Y}$ denotes the (Minkowski) sum of $\mathcal{X}$ and $\mathcal{Y}$, i.e., $\mathcal{X} + \mathcal{Y} \doteq \{x + y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$. $\mathcal{X} - \mathcal{Y}$ denotes the (Minkowski) difference of $\mathcal{X}$ and $\mathcal{Y}$, i.e., $\mathcal{X} - \mathcal{Y} \doteq \{x - y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$.

We denote by $\mathbb{1}_{\mathcal{C}}(\boldsymbol{x})$ the indicator function of a set $\mathcal{C}$, with value 1 if $\boldsymbol{x} \in \mathcal{C}$ and 0 otherwise. Given two vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^n$, we write $\boldsymbol{x} \geq \boldsymbol{y}$ or $\boldsymbol{x} > \boldsymbol{y}$ to mean component-wise $\geq$ or $>$, respectively.

For any set $\mathcal{C}$, we denote the diameter by $\mathrm{diam}(\mathcal{C}) \doteq \max_{\boldsymbol{c}, \boldsymbol{c}' \in \mathcal{C}} \|\boldsymbol{c} - \boldsymbol{c}'\|$.

We define the gradient operator $\nabla_{\boldsymbol{x}}$ as the operator which takes as input a function $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$, and outputs a vector-valued function consisting of the partial derivatives of f w.r.t. $\boldsymbol{x}$. We define the subdifferential of any function $f : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ w.r.t. variable $\boldsymbol{x}$ at a point $(\boldsymbol{a}, \boldsymbol{b}) \in \mathcal{X} \times \mathcal{Y}$ by $\mathcal{D}_{\boldsymbol{x}} f(\boldsymbol{a}, \boldsymbol{b}) \doteq \{\boldsymbol{h} \mid f(\boldsymbol{x}, \boldsymbol{b}) \geq f(\boldsymbol{a}, \boldsymbol{b}) + \boldsymbol{h}^T(\boldsymbol{x} - \boldsymbol{a})\}$, and we denote the derivative operator (resp. partial derivative operator w.r.t. $\boldsymbol{x}$) of any function $\boldsymbol{g} : \mathcal{X} \times \mathcal{Y} \to \mathcal{Z}$ by $\partial \boldsymbol{g}$ (resp. $\partial_{\boldsymbol{x}} \boldsymbol{g}$).

Functions. Given a Euclidean vector space $\mathcal{X} \subseteq \mathbb{R}^n$, we define its dual space $\mathcal{X}^*$ as the set of all linear maps $\boldsymbol{f} : \mathcal{X} \to \mathbb{R}^d$. Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$, $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be normed spaces. Consider a function $\boldsymbol{f} : \mathcal{X} \to \mathcal{Y}$. $\boldsymbol{f}$ is **continuous** if for all sequences $\{x^{(n)}\}_{n \in \mathbb{N}}$ s.t. $x^{(n)} \to x \in \mathcal{X}$, we have $f(x^{(n)}) \to f(x)$. Given $\ell \geq 0$, $\boldsymbol{f}$ is said to be ℓ-**Lipschitz-continuous** on $\mathcal{A} \subseteq \mathcal{X}$ iff $\forall \boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathcal{A}, \|f(\boldsymbol{x}_1) - f(\boldsymbol{x}_2)\|_{\mathcal{Y}} \leq \ell \|\boldsymbol{x}_1 - \boldsymbol{x}_2\|_{\mathcal{X}}$. Consider a function $f : \mathcal{X} \to \mathbb{R}$. f is **convex** iff for all $\lambda \in [0, 1]$ and $\boldsymbol{x}, \boldsymbol{x}' \in \mathcal{X}$, $f(\lambda \boldsymbol{x} + (1-\lambda)\boldsymbol{x}') \leq \lambda f(\boldsymbol{x}) + (1-\lambda) f(\boldsymbol{x}')$. Given $\mu \geq 0$, f is μ-strongly-convex, iff $\boldsymbol{x} \mapsto f(\boldsymbol{x}) - \mu/2 \|\boldsymbol{x}\|^2$ is convex.

Correspondences. Let $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ be an inner product space. Consider a correspondence $\mathcal{R} : \mathcal{X} \rightrightarrows \mathcal{X}^*$. $\mathcal{R}$ is said to be **upper hemicontinuous** if for any sequence $\{(x^{(n)}, y^{(n)}\}_{n \in \mathbb{N}_+} \subset \mathcal{X} \times \mathcal{X}^*$ such that $(x^{(n)}, y^{(n)}) \to (x, y)$ and $y^{(n)} \in \mathcal{R}(x^{(n)})$ for all $n \in \mathbb{N}_+$, it also holds that $y \in \mathcal{R}(x)$. A correspondence $\mathcal{R}$ is **continuous** if for any sequence $\{x^{(n)}\}_{n \in \mathbb{N}_+} \subset \mathcal{X}$ such that $x^{(n)} \to x$, we have $\mathcal{R}(x^{(n)}) \to \mathcal{R}(x)$. A correspondence is said to be **closed-valued** (resp. **compact-valued** / **convex-valued** / **singleton-valued**) iff for all $x \in \mathcal{X}$, $\mathcal{R}(x)$ is closed (resp. compact / convex / a singleton). $\mathcal{R}$ is **monotone** iff for all $x, x' \in \mathcal{X}$, and $y \in \mathcal{R}(x), y' \in \mathcal{R}(x')$, $\langle y' - y, x' - x \rangle \geq 0$. $\mathcal{R}$ is **pseudomonotone** iff for all $x, x' \in \mathcal{X}$, and $y \in \mathcal{R}(x), y' \in \mathcal{R}(x')$, $\langle y', x' - x \rangle \geq 0 \implies \langle y, x - x' \rangle \geq 0$. $\mathcal{R}$ is **quasimonotone** iff for all $x, x' \in \mathcal{X}$, and $y \in \mathcal{R}(x), y' \in \mathcal{R}(x')$, $\langle y', x' - x \rangle > 0 \implies \langle y, x - x' \rangle \geq 0$. We note the following relationship between these notions of monotonicity: $\text{monotone} \implies \text{pseudomonotone} \implies \text{quasimonotone}$.

2.2 Set Theory

2.2.1 Sets

The notion of a **set** (or collection or family) which are objects that comprise of **elements** (or **points**) is taken as a primitive throughout, and as such a background in set theory is prerequisite for understanding the notions developed in this thesis. We refer the reader to the prologue of Folland (1999) for the necessary background.

We use the shorthands $\forall$ and $\exists$ to respectively mean for all (or for every), and there exists (or for some).

Unless otherwise noted, letters will be used as variables, i.e., placeholders that denote a value. We use caligraphic uppercase letters or Greek uppercase letters to denote sets (e.g., $\mathcal{X}$ or Φ).

$\mathcal{X} = \{u, v, w\}$ denotes the elements of the set $\mathcal{X}$, namely u, v, w. Order and repetitions are insignificant, that is, $\{u, v, w\} = \{u, w, w, v, u, w, v\}$.

The elipsis $\ldots$ is meant to be understood as a logical completion of any sequence, e.g., $\mathcal{X} = \{u, v, w, \ldots\} = \{u, v, w, x, y, z\}$.

$\emptyset \doteq \{\}$ denotes the empty set $\{\}$, i.e., the set without any element.

$\mathcal{X}$ is called a **singleton** iff it contains only one element.

$[n]$ denotes the set of integers $\{1, \ldots, n\}$ by

$[n : m]$ denotes the set of integers $\{n, n + 1 \ldots, m\}$ by

$\mathbb{N}$ denotes the set of natural numbers $\{1, 2, 3, \ldots\}$

$\mathbb{R}$ denotes the set of real numbers (i.e., the set of all numbers strictly between negative infinity $-\infty$ and ∞).

$\bar{\mathbb{R}}$ denotes the set of extended real numbers $\mathbb{R} \cup \{-\infty, \infty\}$.

We denote the positive and strictly positive (resp. negative and strictly negative) elements of a set by $+$ and $++$ (resp. $-$ and $--$) subscripts, respectively, e.g., $\mathbb{R}_+$ and $\mathbb{R}_{--}$.

$x \in \mathcal{X}$ means x is an element of the set $\mathcal{X}$.

$x \notin \mathcal{X}$ means x is not an element of the set $\mathcal{X}$.

$\mathcal{Y} \subseteq \mathcal{X}$ (or $\mathcal{X} \supseteq \mathcal{Y}$) means every element of $\mathcal{Y}$ is also an element of the set of $\mathcal{X}$, in which case we will say that $\mathcal{Y}$ is a subset of $\mathcal{X}$.

$\mathcal{Y} = \mathcal{X}$ means that $\mathcal{Y} \subseteq \mathcal{X}$ and $\mathcal{Y} \supseteq \mathcal{X}$

$\mathcal{Y} \subset \mathcal{X}$ ($\mathcal{Y} \subsetneq \mathcal{X}$ or $\mathcal{X} \supset \mathcal{Y}$ or $\mathcal{X} \supsetneq \mathcal{Y}$) means that we have $\mathcal{Y} \subseteq \mathcal{X}$ but not $\mathcal{Y} = \mathcal{X}$, in which case we will say that $\mathcal{Y}$ is a strict subset of $\mathcal{X}$.

A set $\mathcal{X}$ is **non-empty** iff $\mathcal{X} \neq \emptyset$.

A set $\mathcal{X}$ is said to be **affine** iff for all $\lambda \in \mathbb{R}$ and $x, y \in \mathcal{X}$, we have $\lambda x + (1 - \lambda) y \in \mathcal{X}$.

A set $\mathcal{X}$ is said to be **convex** iff for all $\lambda \in [0, 1]$ and $x, y \in \mathcal{X}$, we have $\lambda x + (1 - \lambda) y \in \mathcal{X}$.

Given any set $\mathcal{X} \subseteq \mathcal{U}$, its **power set** the collection of all of its subsets by $2^{\mathcal{X}} \doteq \{\mathcal{Y} \subseteq \mathcal{X}\}$.

The set of all elements, which will be clear from context, is called the **universal set**, and is often denoted by $\mathcal{U}$.

$\{x \in \mathcal{X} \mid P(x)\}$ denotes the set of all elements $x \in \mathcal{X}$ for which the proposition $P(x)$ is true.

$\mathcal{X} \cup \mathcal{Y}$ is called the **union** of the sets $\mathcal{X}$ and $\mathcal{Y}$, and is given by the set $\{x \in \mathcal{U} \mid x \in \mathcal{X} \text{ or } x \in \mathcal{Y}\}$.

$\mathcal{X} \cap \mathcal{Y}$ is called the **intersection** of the sets $\mathcal{X}$ and $\mathcal{Y}$, and is given by the set $\{x \in \mathcal{U} \mid x \in \mathcal{X} \text{ and } x \in \mathcal{Y}\}$. We will often write $\{x \mid x \in \mathcal{X}, x \in \mathcal{Y}\}$ to mean $\{x \in \mathcal{U} \mid x \in \mathcal{X} \text{ and } x \in \mathcal{Y}\}$.

Two sets $\mathcal{X}, \mathcal{Y} \subseteq \mathcal{U}$ are said to be **disjoint** iff their intersection is empty, i.e., $\mathcal{X} \cap \mathcal{Y} = \emptyset$. A collection of sets $\mathcal{E} \subseteq \mathcal{U}$ is said to be **pairwise disjoint** iff for all $\mathcal{X}, \mathcal{Y} \in \mathcal{E}$ s.t. $\mathcal{X} \neq \mathcal{Y}$, $\mathcal{X}$ and $\mathcal{Y}$ are disjoint

If $\mathcal{E} \subseteq \mathcal{U}$ is a collection of sets, we define the the union and intersection of its members respectively as:

$$\bigcup_{\mathcal{X} \in \mathcal{E}} \mathcal{X} \doteq \{x \in \mathcal{U} \mid \text{ for some } x \in \mathcal{X}\}$$

$$\bigcap_{\mathcal{X} \in \mathcal{E}} \mathcal{X} \doteq \{x \in \mathcal{U} \mid \text{ for all } x \in \mathcal{X}\}$$

$\{\mathcal{X}_i\}_{i\in\mathcal{I}}$ denotes the collection of the $\mathcal{X}_i$'s, i.e., $\bigcup_{i\in\mathcal{I}}\{\mathcal{X}_i\}$. Similarly, for any $k, n \in \mathbb{N}$ s.t. $k \geq n$, $\{\mathcal{X}_i\}_{i=k}^n$ denotes the collection of the sets $\mathcal{X}_k, \ldots, \mathcal{X}_n$, i.e., $\bigcup_{i=k}^n\{\mathcal{X}_i\}$. When clear from context, we will often write $\{\mathcal{X}_i\}_i$.

For any set $\mathcal{X} \subseteq \mathcal{M}$, and family of sets $\{\mathcal{Y}^{(i)}\}_{i\in\mathcal{I}}$ such that $\mathcal{X} \subseteq \bigcup_{i\in\mathcal{I}}\mathcal{Y}^{(i)}$, $\{\mathcal{Y}^{(i)}\}_{i\in\mathcal{I}}$ is called a **cover** of $\mathcal{X}$, and $\mathcal{X}$ is said to be **covered by** the $\mathcal{Y}^{(i)}$'s.

$\mathcal{X}\backslash\mathcal{Y}$ is called the **difference** of the sets $\mathcal{X}$ and $\mathcal{Y}$, and is given by $\{x \in \mathcal{U} \mid x \in \mathcal{X} \text{ and } x \notin \mathcal{Y}\}$.

The **complement** $\mathcal{X}^c$ of a set $\mathcal{X} \subseteq \mathcal{U}$ is given by the difference of the universal set $\mathcal{U}$ and $\mathcal{X}$, i.e., $\mathcal{X}^c \doteq \mathcal{U}\backslash\mathcal{X}$.

A n-**tuple** (or tuple when clear from context) $\boldsymbol{x} \doteq (x_1, x_2, \ldots, x_n)$ is an ordered array of $n \in \mathbb{N}_{++}$ elements s.t. $(x_1, x_2, \ldots, x_n) = (y_1, y_2, \ldots, y_n)$ iff $x_i = y_i$ for all $i \in [n]$. We stress that a variable is a tuple by denoting it with a bold lowercase letters, but in general a normal font lowercase letter can also represent a tuple. $(x_i, \boldsymbol{x}_{-i})$ denotes the tuple $\boldsymbol{x} \doteq (x_1, \ldots, x_n)$, where $\boldsymbol{x}_{-i}$ denotes $\boldsymbol{x}$ with the ith element $\boldsymbol{x}$ removed. A 2-tuple is often also called a **pair**. For any $k, n \in \mathbb{N}$ s.t. $k \geq n$, $\{x_i\}_{i=k}^n$ denotes the tuple $(x_k, \ldots, x_n)$. When clear from context, we will often write $(x_i)_i$.

For convenience, we will denote a n-tuple which consists of the same number by the number in bold font with a subscript of n, e.g., $\mathbf{0}_3 \doteq (0, 0, 0)$ or $\mathbf{1}_4 \doteq (1, 1, 1, 1)$. When clear from, context, we will often omit the subscript.

We denote the n-dimensional ith **basis vector** $\boldsymbol{j}_i \doteq (1, \mathbf{0}_{n,-i})$

$\mathcal{X} \times \mathcal{Y}$ is the **Cartesian product** of sets $\mathcal{X}$ and $\mathcal{Y}$ and is given by $\{(x, y) \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$. The Cartesian product of a collection $\{\mathcal{X}_i\}_{i\in[n]}$ of $n \in \mathbb{N}$ sets is given by $\bigtimes_{i\in[n]}\mathcal{X}_i \doteq \mathcal{X}_1 \times \ldots \times \mathcal{X}_n$. When for all $i \in [n]$, $\mathcal{X}_i = \mathcal{Y}$, we write $\mathcal{Y}^n \doteq \bigtimes_{i\in[n]}\mathcal{X}_i$.

Δ_n denotes the unit simplex in $\mathbb{R}^n$, i.e., $\Delta_n = \{\boldsymbol{x} \in \mathbb{R}_+^n \mid \sum_{i=1}^n x_i = 1\}$.

We denote the **affine hull** of any set $\mathcal{X}$ by $\text{aff}(\mathcal{X}) \doteq \bigcap_{\mathcal{Y}\supset\mathcal{X}:\mathcal{Y} \text{ is affine}}\mathcal{Y}$. For $\mathcal{X} \subseteq \mathbb{R}^n$, by Caratheodory's theorem, this definition reduces to $\text{conv}(\mathcal{X}) \doteq \{\sum_{i=1}^{n+1}\lambda_i x_i \mid \forall\boldsymbol{\lambda} \in \mathbb{R}^{n+1} \text{ s.t. } \sum_{i\in[n+1]}\lambda_i = 1\}$. We denote the **convex hull** of any set $\mathcal{X}$ by $\text{conv}(\mathcal{X}) \doteq \bigcap_{\mathcal{Y}\supset\mathcal{X}:\mathcal{Y} \text{ is convex}}\mathcal{Y}$. For $\mathcal{X} \subseteq \mathbb{R}^n$, by Caratheodory's theorem, this definition reduces to $\text{conv}(\mathcal{X}) \doteq \{\sum_{i=1}^{n+1}\lambda_i x_i \mid \forall\boldsymbol{\lambda} \in \Delta_{n+1}\}$.

For any real number $a \in \mathbb{R}$, $a\mathcal{X}$ denotes the (Minkowski) product, i.e., $a\mathcal{Y} \doteq \{ax \mid x \in \mathcal{X}\}$.

$\mathcal{X} + \mathcal{Y}$ denotes the (Minkowski) sum of $\mathcal{X}$ and $\mathcal{Y}$, i.e., $\mathcal{X} + \mathcal{Y} \doteq \{x + y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$.

$\mathcal{X} - \mathcal{Y}$ denotes the (Minkowski) difference of $\mathcal{X}$ and $\mathcal{Y}$, i.e., $\mathcal{X} - \mathcal{Y} \doteq \{x - y \mid x \in \mathcal{X}, y \in \mathcal{Y}\}$.

2.2.2 Relations

Relations Given sets $\mathcal{X}$ and $\mathcal{Y}$, a **relation** $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$ from $\mathcal{X}$ to its **codomain** $\mathcal{Y}$ is a subset of $\mathcal{X} \times \mathcal{Y}$. For any $x \in \mathcal{X}, y \in \mathcal{Y}$, we define $x \succeq_{\mathcal{R}} y$ to mean $(x, y) \in \mathcal{R}$.

The **domain** $\mathrm{dom}(\mathcal{R})$ of a relation $\mathcal{R}$ is given by the set $\mathrm{dom}(\mathcal{R}) \doteq \{x \in \mathcal{X} \mid x \succeq_{\mathcal{R}} y, \exists y \in \mathcal{Y}\}$.

The **range** (or **image**) of a relation $\mathcal{R}$ is given by the set $\mathrm{range}(\mathcal{R}) \doteq \{y \in \mathcal{Y} \mid x \succeq y, \exists x \in \mathcal{X}\}$.

The **inverse** $\mathcal{R}^{-1} \subseteq \mathcal{Y} \times \mathcal{X}$ of a relation $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$ is $\mathcal{R}^{-1} \doteq \{(y, x) \in \mathcal{Y} \times \mathcal{X} \mid (x, y) \in \mathcal{R}\}$.

The **image** $\mathcal{R}(x) \subseteq \mathcal{Y}$ of $x \in \mathcal{X}$ under a relation $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$ is $\mathcal{R}(x) \doteq \{y \in \mathcal{Y} \mid x \succeq_{\mathcal{R}} y\}$.

The **image** $\mathcal{R}(\mathcal{A}) \subseteq \mathcal{Y}$ of a set $\mathcal{A} \subseteq \mathcal{X}$ under a relation $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$ is $\mathcal{R}(\mathcal{A}) \doteq \bigcup_{x \in \mathcal{A}} \mathcal{R}(x)$.

Let $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$ be a relation from $\mathcal{X}$ to $\mathcal{Y}$, and $\mathcal{R}'\mathcal{Y} \times \mathcal{Z}$ be a relation from $\mathcal{Y}$ to $\mathcal{Z}$. The **composition** $\mathcal{R}' \circ \mathcal{R}$ of $\mathcal{R}'$ with $\mathcal{R}$ is defined as:

$$\mathcal{R}' \circ \mathcal{R} \doteq \{(x, z) \in \mathcal{X} \times \mathcal{Z} \mid \exists y \in \mathcal{Y} \text{ s.t. } x \succeq_{\mathcal{R}} y, y \succeq_{\mathcal{R}'} z\}$$

If $\mathcal{Y} \doteq \mathcal{X}$, then $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{X}$ is said to be a **(binary) relation** on $\mathcal{X}$, in which case for all $x, y \in \mathcal{X}$, we say that x **succeeds** y iff $x \succeq_{\mathcal{R}} y$. Further, we define the following. We say that x **preceeds** y and denote $x \preceq_{\mathcal{R}} y$ to mean $(y, x) \in \mathcal{R}$. We say that x is **similar** to y, and denote $x \simeq y$ iff $x \succeq_{\mathcal{R}} y$ and $y \preceq_{\mathcal{R}} x$, $x \not\simeq y$ if otherwise. We say that x **strictly succeeds** (resp. **strictly preceeds**) y and denote $x \succ_{\mathcal{R}} y$ (resp. $x \prec_{\mathcal{R}} y$) to mean $x \succeq_{\mathcal{R}} y$ and $x \not\simeq_{\mathcal{R}} y$ (resp. $x \preceq_{\mathcal{R}} y$ and $x \not\simeq_{\mathcal{R}} y$). When $\mathcal{R}$ is clear from context, we will denote $\succeq_{\mathcal{R}}$, $\preceq_{\mathcal{R}}$, and $\simeq_{\mathcal{R}}$ by $\succeq$, $\preceq$, and $\simeq$ respectively.

For any binary relation $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{X}$, we also define the following properties. The relation $\mathcal{R}$ is **complete** iff for all $x, y \in \mathcal{X}$, either $x \succeq y$ or $x \preceq y$, or $x \simeq y$. The relation $\mathcal{R}$ is **transitive** iff, for all $x, y, \boldsymbol{z} \in \mathcal{X}$, $x \succeq \boldsymbol{z}$ whenever $x \succeq y$ and $y \succeq \boldsymbol{z}$. The relation $\mathcal{R}$ is **antisymmetric** iff for all $x, y \in \mathcal{X}$, $x \succeq y$ and $x \preceq y$, then $x = y$. The relation $\mathcal{R}$ is **reflexive** iff for all $x \in \mathcal{X}$, $x \succeq x$.

A **partial order** $(\mathcal{U}, \mathcal{R})$ consists of a unviersal set $\mathcal{U}$ and a binary relation $\mathcal{R} \subseteq \mathcal{U} \times \mathcal{U}$ which is transitive, antisymmetric, and reflexive. If, in addition, $\mathcal{R}$ is complete, then $(\mathcal{U}, \mathcal{R})$ is complete order. When $\mathcal{U} \doteq \mathbb{R}$, then we will assume that $(\mathcal{U}, \mathcal{R})$ is the usual (total) order on $\mathbb{R}$, in which case we denote $\succeq_\mathcal{R}, \preceq_\mathcal{R}, \simeq_\mathcal{R}, \succ_\mathcal{R}, \prec_\mathcal{R}$ by $\geq, \leq, =, >, <$. When $\mathcal{U} \doteq \mathbb{R}^n$, then we will assume the partial order $(\mathcal{U}, \mathcal{R})$ defined by the relation $\mathcal{R} \doteq \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^n \mid x_i \geq y_i, \forall i \in [n]\}$, in which case, overloading notation, we denote $\succeq_\mathcal{R}, \preceq_\mathcal{R}, \simeq_\mathcal{R}, \succ_\mathcal{R}, \prec_\mathcal{R}$ by $\geq, \leq, =, >, <$.

For any partially ordered set $(\mathcal{U}, \mathcal{R})$, we define the **infimum** or **lower bound** (resp. **supremum** or **upper bound**) of a set $\mathcal{X}$ as an element $x^* \in \mathcal{U}$ s.t. for all $x \in \mathcal{X}$, $x \succeq x^*$ (resp. $x \preceq x^*$), in which case we denote $\inf(\mathcal{X}) \doteq x^*$ (resp. $\sup(\mathcal{X}) \doteq x^*$). If the infimum $\inf(\mathcal{X})$ (resp. supremum $\sup(\mathcal{X})$) is an element of $\mathcal{X}$, we then write $\min(\mathcal{X})$ (resp. $\max(\mathcal{X})$).

2.2.3 Correspondences

A relation $\mathcal{R} \subset \mathcal{X} \times \mathcal{Y}$ from $\mathcal{X}$ to $\mathcal{Y}$ s.t. $\operatorname{dom}(\mathcal{R}) = \mathcal{X}$ is called a **correspondence** from $\mathcal{X}$ to its **codomain** $\mathcal{Y}$ and denoted $\mathcal{R} : \mathcal{X} \rightrightarrows \mathcal{Y}$. A **correspondence** $\mathcal{R} : \mathcal{X} \rightrightarrows \mathcal{Y}$ can be understood as a map[1] from $\mathcal{X}$ to subsets of $\mathcal{Y}$, in which for all $x \in \mathcal{X}$ we write $x \mapsto\mathrel{\mkern-14mu}\rightrightarrows \mathcal{R}(x) \subseteq \mathcal{Y}$. As a correspondence is a relation (i.e., a set) throughout this thesis, we will denote correspondences by a caligraphic uppercase letter (e.g., $\mathcal{R}$). A correspondence is said to be **non-empty-valued** (resp. **convex-valued**) iff for all $x \in \mathcal{X}$, $\mathcal{R}(x)$ is **non-empty** (resp. **convex**).

2.2.4 Functions

A relation $f \subseteq \mathcal{X} \times \mathcal{Y}$ from $\mathcal{X}$ to $\mathcal{Y}$ with the property that for every $x \in \mathcal{X}$, there is a unique $y \in \mathcal{Y}$ s.t. $x \succeq_\mathcal{R} y$ is called a **function (or mapping)** from $\mathcal{X}$ to its **codomain** $\mathcal{Y}$ and denoted $f : \mathcal{X} \to \mathcal{Y}$. We call the element of y the value of f at x, and by abuse of notation, define $x \mapsto f(x) \doteq y$ Thus, while the value of f at x (i.e., y), and its image at x (i.e., $\{y\}$) are both denoted by $f(x)$, the meaning of $f(x)$ will be clear from context. While a function $f : \mathcal{X} \to \mathcal{Y}$ should be understood as a relation (i.e., a set), we will often be working with the value (i.e., an element rather than a set) of f rather than its image $f(x)$, and as such we will often denote functions by a lowercase letter. In some cases, if we want to stress the type of the value of the function, we will denote

[1] When understood as a point-to-set (or multivalued) map, a **correspondence** $\mathcal{R} : \mathcal{X} \rightrightarrows \mathcal{Y}$ is denoted $\mathcal{R} : \mathcal{X} \to 2^\mathcal{Y}$ s.t. for all $x \in \mathcal{X}$, $\mathcal{R}$ maps x to subsets $\mathcal{Y}$, i.e., $x \mapsto\mathrel{\mkern-14mu}\rightrightarrows \mathcal{R}(x) \subseteq \mathcal{Y}$. While some authors have with authority argued the ill-posedness of such a definition (see page 1 of Dieudonné (1960)) such a view can be helpful in obtaining many theoretical results.

the function by the type of its value. For instance, if the function is denoted by a bold lowercase letter (e.g., $\boldsymbol{f}$), then the mapping is vector valued, and if the function is denoted by bold uppercase letter (e.g., $\boldsymbol{F}$), then the mapping is matrix valued.

A function $f : \mathcal{X} \to \mathcal{Y}$ is said to be **affine** (or **linear**[2]) if for all $\alpha, \beta \in \mathbb{R}$ and $x, x' \in \mathcal{X}$ we have $f(\alpha x + \beta x') = \alpha f(x) + \beta f(x')$. We denote by $\mathbb{1}_{\mathcal{X}}(\boldsymbol{x})$ the indicator function (or Dirac delta measure) for a set $\mathcal{X}$, with value 1 if $\boldsymbol{x} \in \mathcal{X}$ and 0 otherwise. We denote by $\chi_{\mathcal{X}}(\boldsymbol{x})$ the **characteristic function** for a set $\mathcal{X}$, with value 0 if $\boldsymbol{x} \in \mathcal{X}$ and ∞ otherwise.

2.2.5 Cardinality

Consider a function $f : \mathcal{X} \to \mathcal{Y}$ from $\mathcal{X}$ to $\mathcal{Y}$.

f is called **injective** (or **one-to-one**) iff $f(x) = f(\boldsymbol{x})$ implies $x = y$.

f is called **surjective** (or **onto**) iff $\mathrm{range}(f) = \mathcal{Y}$.

f is called **bijective** iff it is injective and surjective.

If $\mathcal{X}$ and $\mathcal{Y}$ are non-empty sets, we define the following expressions:

$$\mathrm{card}(\mathcal{X}) \leq \mathrm{card}(\mathcal{Y}) \quad \mathrm{card}(\mathcal{X}) \geq \mathrm{card}(\mathcal{Y}) \quad \mathrm{card}(\mathcal{Y}) = \mathrm{card}(\mathcal{X})$$

to mean that there exists $f : \mathcal{X} \to \mathcal{Y}$ which is injective, surjective or bijective respectively.

Additionally, we define $\mathrm{card}(\mathcal{X}) < \mathrm{card}(\mathcal{Y})$ (or $\mathrm{card}(\mathcal{Y}) > \mathrm{card}(\mathcal{X})$) to mean $\mathrm{card}(\mathcal{X}) \leq \mathrm{card}(\mathcal{Y})$ but not $\mathrm{card}(\mathcal{X}) = \mathrm{card}(\mathcal{Y})$.

Consider a set $\mathcal{X}$. $\mathcal{X}$ is **countable** iff $\mathrm{card}(\mathcal{X}) \leq \mathrm{card}(\mathbb{N})$, **infinite** iff $\mathrm{card}(\mathcal{X}) \geq \mathrm{card}(\mathbb{N})$. $\mathcal{X}$ is **countably infinite** iff it is countable and infinite. A set $\mathcal{X}$ is said to be **finite** iff $\mathrm{card}(\mathcal{X}) < \mathrm{card}(\mathbb{N})$. A finite set $\mathcal{X}$ is said to have cardinality n iff $\mathrm{card}(\mathcal{X}) = \mathrm{card}([n])$, in which case we write $\mathrm{card}(\mathcal{X}) \doteq n$.[3]

[2] We will not be making a distinction between the two concepts.

[3] While this notation might suggest that $\mathrm{card}(\cdot)$ is a function, such an interpretation is only appropriate for finite sets for which cardinality can be interpreted as the number of elements in the the set, and should be avoided for infinite set

2.2.6 Sequences

A function $f : \mathbb{N}_{++} \to \mathcal{X}$ from the set of positive integers $\mathbb{N}_{++}$ to a set $\mathcal{X}$ is called a **sequence** of points in $\mathcal{X}$, and denoted $\{x^{(n)}\}_n \doteq \{f(n)\}_n$ or $(x^{(n)})_n \doteq (f(n))_n$. While we denote a sequence as a tuples or collection, it should always be understood as a function. We will often denote elements of a sequence by superscripts with round brackets, e.g, $x^{(n)}$, to stress the sequential aspect of sequences, in general, a sequence can be denoted by lower or upper scripts without brackets, e.g., x_n and x^n.

2.2.7 Infinite Carthesian Product

If $\{\mathcal{X}_i\}_{i\in\mathcal{I}}$ is an infinite collection of sets (i.e., $\mathcal{I}$ is infinite), then the **Carthesian product** of the $\mathcal{X}_i$ is defined as:

$$\bigtimes_{i\in\mathcal{I}} \mathcal{X}_i \doteq \left\{ f : \mathcal{I} \to \bigcup_{i\in\mathcal{I}} \mathcal{X}_i \right\} \tag{2.1}$$

We note that this definition of the Carthesian product for infinite collections of sets does not agree with the definition of the Carthesian product for finite collections of sets. As such, while for simplicity we will use the same notation convention for both, the definition of the operator $\bigtimes_{i\in\mathcal{I}}$ for infinite $\mathcal{I}$ should be understood as distinct from the one for finite $\mathcal{I}$. This distinction will be clear from context throughout this thesis. With this definition in hand, when for all $i \in \mathcal{I}$, $\mathcal{X}_i = \mathcal{Y}$, notice that $\bigtimes_{i\in\mathcal{I}} \mathcal{X}_i = \bigtimes_{i\in\mathcal{I}} \mathcal{Y} = \mathcal{Y}^{\mathcal{X}} = \{f : \mathcal{X} \to \mathcal{Y}\}$. As such, we denote $\mathcal{Y}^{\mathcal{X}} \doteq \{f : \mathcal{X} \to \mathcal{Y}\}$ the set of all functions from $\mathcal{X}$ to $\mathcal{Y}$.

2.3 Metric Spaces

A **metric space** is a tuple $(\mathcal{M}, d)$ that consists of a set $\mathcal{M}$, and a function $d : \mathcal{M} \times \mathcal{M} \to \mathbb{R}_+$ called a **metric** which takes as input any two points in $\mathcal{M}$ and outputs a value called the **distance** between those two points such that the following hold:

1. (Non-Degeneracy) $d(x, y) = 0 \iff x = y, \forall x, y \in \mathcal{M}$
2. (Triangle Inequality) $d(x, z) \leq d(x, y) + d(y, z), \forall x, y, z \in \mathcal{M}$
3. (Symmetry) $d(x, y) = d(y, x), \forall x, y \in \mathcal{M}$

A sequence $\{x^{(n)}\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ is said to **converge** to some $x \in \mathcal{M}$ if for every $\varepsilon > 0$, there exists an integer $\overline{n} \in \mathbb{N}$ s.t. for all integers $m \geq \overline{n}$ we have that $d(x^{(m)}, x) \leq \varepsilon$. When the metric space is clear from context, if the sequence $\{x^{(n)}\}_{n\in\mathbb{N}} \subseteq \mathcal{M}$ converges to $x \in \mathcal{M}$, we then write $\lim_{n\to\infty} x^{(n)} = x$ or $x^{(n)} \to x$. A sequence $\{x^{(n)}\}_{n\in\mathbb{N}} \subset \mathcal{M}$ is said to be a **Cauchy sequence** if for all $\epsilon > 0$, there exists $\overline{n} \in \mathbb{N}$ such that for all integers $n, m > \overline{n}$, we have $d(x^{(n)}, x^{(m)}) < \epsilon$.

A set $\mathcal{X}$ is said to be **closed** if for any convergent sequence $\{x^{(n)}\}_{n\in\mathbb{N}} \subseteq \mathcal{X}$ s.t. $x^{(n)} \to x \in \mathcal{M}$, we have $x \in \mathcal{X}$. A set $\mathcal{X}$ is said to be **open** if its complement $\mathcal{M} \setminus \mathcal{X}$ is closed. For any set $\mathcal{X}$, we define the distance of any point x to the set $\mathcal{X}$ as $d(x, \mathcal{X}) \doteq \min_{x'\in\mathcal{X}} d(x, x')$. We also define the diameter of a set by $\text{diam}(\mathcal{X}) \doteq \max_{x,x'\in\mathcal{X}} d(x, x')$. A set $\mathcal{X}$ is said to be **bounded** iff $\text{diam}(\mathcal{X}) < \infty$. A set

Throughout this thesis, we will be dealing with complete metric spaces. A subset $\mathcal{X} \subseteq \mathcal{M}$ of $\mathcal{M}$ is said to be **complete** if any Cauchy sequence $\{x^{(n)}\}_{n\in\mathbb{N}} \subset \mathcal{X}$ in $\mathcal{X}$ converges, i.e., $x^{(n)} \to x \in \mathcal{M}$, and its limit is in $\mathcal{X}$ i.e., $x \in \mathcal{X}$. A metric space $(\mathcal{M}, d)$ is said to be **complete** if $\mathcal{M}$ is complete. We note that any closed subset of a complete metric space is complete. A common example of a complete metric space that we will deal throughout this thesis is the **Euclidean (metric) space** $(\mathbb{R}^n, d)$ with the **Euclidean metric** $d(\boldsymbol{x}, \boldsymbol{y}) \doteq \sqrt{\sum_{i=1}^{n}(x_i - y_i)^2}$.

For any $\varepsilon \geq 0$, we write $\mathcal{B}_\varepsilon[x] = \{x' \in \mathcal{M} \mid d(x, x') \leq \varepsilon\}$ and $\mathcal{B}_\varepsilon(x) = \{x' \in \mathcal{M} \mid d(x, x') \leq \varepsilon\}$ to respectively denote the closed and open ε-ball centered at $x \in \mathcal{M}$. A point $x \in \mathcal{X}$ is called an interior point of $\mathcal{X}$, if there exists $\varepsilon > 0$ s.t. $\mathcal{B}_\varepsilon(x) \subseteq \mathcal{X}$. The **interior** $\text{int}(\mathcal{X})$ of a set $\mathcal{X}$ consists of the set of all interior points of $\mathcal{X}$. The **relative interior** of a set $\mathcal{X}$ consists of the interior of $\mathcal{X}$ within its affine hull, i.e.,

$\text{relint}(\mathcal{X}) \doteq \{x \in \mathcal{X} \mid \exists \varepsilon > 0, \mathcal{B}_\varepsilon(x) \cap \text{aff}(\mathcal{X}) \subseteq \mathcal{X}\}$ A set $\mathcal{X} \subseteq \mathcal{M}$ is said to be **totally bounded** if for every $\varepsilon > 0$, it can be covered by finitely many open ε-balls. A set $\mathcal{X} \subseteq \mathcal{M}$ is said to be **compact** if it is complete and totally bounded. By the Heine-Borel Theorem, any closed and bounded subset of $\mathbb{R}^n$ is compact.

A relation $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$ is **upper hemicontinuous (or outer**[4]) hemicontinuous if for any sequence $\{(x^{(n)}, y^{(n)})\}_{n \in \mathbb{N}_+} \subset \mathcal{X} \times \mathcal{Y}$ such that $(x^{(n)}, y^{(n)}) \to (x, y)$ and $x^{(n)} \succeq y^{(n)}$ for all $n \in \mathbb{N}_+$, it also holds that $x \succeq y$. Note that if $\mathcal{R}$ is a compact set, then it is upper hemicontinuous. A relation $\mathcal{R} \subseteq \mathcal{X} \times \mathcal{Y}$ is **inner (or lower) hemicontinuous** if for any $y \in \mathcal{Y}$ and sequence $\{x^{(n)}\}_{n \in \mathbb{N}_+} \subseteq \mathcal{X}$ such that $x^{(n)} \to x$ and $x \succeq y$, there exists $\{y^{(n)}\}_{n \in \mathbb{N}_+} \subseteq \mathcal{Y}$ s.t. $x^{(n)} \succeq y^{(n)}$ for all $n \in \mathbb{N}_+$ and $y^{(n)} \to y$. A relation is said to be continuous if it is both upper and lower hemicontinuous, or equivalently if for any sequence $\{(x^{(n)}, y^{(n)})\}_{n \in \mathbb{N}_+} \subset \mathcal{X} \times \mathcal{Y}$ such that $x^{(n)} \to x$ and $x^{(n)} \succeq y^{(n)}$ for all $n \in \mathbb{N}_+$, it also holds that $y^{(n)} \to y$ and $x \succeq y$.

Since any correspondence is a relation, we can define analogous definition of (upper/lower) hemicontinuity. A correspondence $\mathcal{R} : \mathcal{X} \rightrightarrows \mathcal{Y}$ is said to be **upper (or outer) hemicontinuous** if for any sequence $\{(x^{(n)}, y^{(n)}\}_{n \in \mathbb{N}_+} \subset \mathcal{X} \times \mathcal{Y}$ such that $(x^{(n)}, y^{(n)}) \to (x, y)$ and $y^{(n)} \in \mathcal{R}(x^{(n)})$ for all $n \in \mathbb{N}_+$, it also holds that $y \in \mathcal{R}(x)$. A correspondence $\mathcal{R} : \mathcal{X} \rightrightarrows \mathcal{Y}$ is said to be **lower (or inner) hemicontinuous** if for any $y \in \mathcal{Y}$ and sequence $\{x^{(n)}\}_{n \in \mathbb{N}_+} \subset \mathcal{X}$ such that $x^{(n)} \to x$ and $y \in \mathcal{R}(x)$, there exists $\{y^{(n)}\}_{n \in \mathbb{N}_+} \subset \mathcal{Y}$ s.t. $y^{(n)} \in \mathcal{R}(x^{(n)})$ for all $n \in \mathbb{N}_+$ and $y^{(n)} \to y$. A correspondence $\mathcal{R} : \mathcal{X} \rightrightarrows \mathcal{Y}$ is **continuous** if for any sequence $\{x^{(n)}\}_{n \in \mathbb{N}_+} \subset \mathcal{X}$ such that $x^{(n)} \to x$, we have $\mathcal{R}(x^{(n)}) \to \mathcal{R}(x)$. A correspondence is said to be **closed-valued** (resp. **compact-valued** / **convex-valued** / **singleton-valued**) iff for all $x \in \mathcal{X}$, $\mathcal{R}(x)$ is closed (resp. compact / convex / a singleton).

For functions, the analogous definitions of upper and lower hemicontinuous relations can be shown be equivalent, and as such considering only continuity becomes enough. In particular, a function f is **continuous** if for all sequences $\{x^{(n)}\}_{n \in \mathbb{N}}$ s.t. $x^{(n)} \to x \in \mathcal{X}$, we have $f(x^{(n)}) \to f(x)$. Equivalently, a function $f : \mathcal{X} \to \mathcal{Y}$ is said to be **continuous** if for all $x \in \mathcal{X}$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y \in \mathcal{X}$ with $d(x, y) < \delta$ implies $d'(f(x), f(y)) < \epsilon$.

[4]Note that certain authors make a distinction between upper and outer hemicontinuity by adopting a weaker definition of outer hemicontinuity (see, for instance Border (2010)). For our purposes, we will assume these two terms to be equivalent.

2.4 Normed Spaces

A **normed (vector) space** is a tuple $(\mathcal{X}, \|\cdot\|)$ that consists of a **(vector) space** $\mathcal{X}$ and a function $\|\cdot\|: \mathcal{X} \to \mathbb{R}_+$ called a **norm** such that the following hold:

1. (Normalized) $\|x\| = 0 \iff x = 0, \forall x \in \mathcal{X}$
2. (Homogeneity) $\|cx\| = |c|\|x\|, \forall c \in \mathbb{R}, x \in \mathcal{X}$
3. (Triangle Inequality) $\|x + y\| \leq \|x\| + ||y||, \forall x, y \in \mathcal{X}$

Note that $\mathcal{X}$ does not have to be a set of tuples, and can be any set that satisfies the axioms of vector spaces (e.g., a set of functions). Any normed space $(\mathcal{X}, \|\cdot\|)$ is a metric space since any norm $\|\cdot\|$ defines the **induced metric** $d(x, y) = \|x - y\|$ such that $(\mathcal{X}, d)$ is a valid metric space called the **induced metric space by** $(\mathcal{X}, \|\cdot\|)$. That is, any normed space is also a metric space, and as such the definitions provided in Section 2.3 all apply to normed spaces.

The canonical example of a normed vector space is the ℓ_n^p normed space $(\mathbb{R}^n, \|\cdot\|_p)$ defined by the p-norm $\|\boldsymbol{x}\|_p \doteq \sqrt[p]{\sum_{i=1}^n x_i^p}$. For $p \to \infty$, we obtain the **uniform** (or **sup**) **norm** $\|\cdot\|_\infty$, which is defined as $\|\boldsymbol{x}\|_\infty \doteq \max_{i \in [n]}\{|x_i|\}$. Throughout this thesis, we will mostly be working with the **Euclidean (normed) space** (or ℓ_n^2 **normed space**) $(\mathbb{R}^n, \|\cdot\|_2)$. If the metric space $(\mathcal{X}, d)$ induced by $(\mathcal{X}, \|\cdot\|)$ is complete, then $(\mathcal{X}, \|\cdot\|)$ is called a **Banach space** (or a **complete normed space**). Note that any ℓ_n^p space is a Banach space.

Let $(\mathcal{X}, \|\cdot\|_\mathcal{X})$ and $(\mathcal{Y}, \|\cdot\|_\mathcal{Y})$ be **normed spaces**. A function $f : (\mathcal{X}, \|\cdot\|_\mathcal{X}) \to (\mathcal{Y}, \|\cdot\|_\mathcal{Y})$ or correspondence $\mathcal{R} : (\mathcal{X}, \|\cdot\|_\mathcal{X}) \rightrightarrows (\mathcal{Y}, \|\cdot\|_\mathcal{Y})$ between normed spaces is called an **operator**; we will often denote $f : \mathcal{X} \to \mathcal{Y}$ and $\mathcal{R} : \mathcal{X} \rightrightarrows \mathcal{Y}$ respectively when clear from context.

A seminal theorem in functional analysis is the **Kakutani-Glicksberg Fixed Point Theorem** which provides sufficient conditions for a **fixed point** of a correspondence $\mathcal{R} : \mathcal{X} \rightrightarrows \mathcal{X}$, i.e., a point $x \in \mathcal{X}$ s.t. $x \in \mathcal{R}(x)$ to exist.[5]

[5]While Kakutani presented his results for Euclidean spaces, subsequently Glicksberg generalized his result to convex Hausdorff linear topological spaces (see Section 1 of Glicksberg (1952) for the relevant definitions). Hence, since any normed (vector) space is a convex Hausdorff linear topological space (see, for instance Section 5 of Folland (1999)), we state his result for the special case of normed spaces which concerns all applications of this theorem in this thesis. Since any Euclidean space is a normed space, the version of the Theorem stated here generalizes Kakutani's to non-Euclidean spaces (e.g., functional spaces). Additionally, note that Glicksberg (1952) states his result for closed correspondences, but as the domain and range of the correspondence is compact-valued, it suffices to assume upper hemicontinuity instead, as any upper hemicontinuous and compact-valued correspondence is closed (see Section 2.3).

Theorem 2.4.1 [Kakutani-Glicksberg Fixed Point Theorem (Kakutani, 1941; Glicksberg, 1952)]**.**
Consider a normed space $(\mathcal{U}, \|\cdot\|)$, and a non-empty, compact, convex set $\mathcal{X} \subseteq \mathcal{U}$. Let $\mathcal{R} : \mathcal{X} \rightrightarrows \mathcal{X}$ be 1) upper hemicontinuous, and 2) non-empty-, compact, and convex-valued. Then, there exists a fixed point $x \in \mathcal{X}$ s.t. $x \in \mathcal{R}(x)$.

2.5 Inner Product Spaces

An **inner product space** is a tuple $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ that consists of a vector space $\mathcal{X}$ and a function $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ called an **inner product** (or **scalar product**) such that the following hold:

1. (Normalized) $\langle x, x \rangle > 0, \forall x \neq 0$
2. (Symmetry) $\langle x, y \rangle = \langle y, x \rangle, \forall x, y \in \mathcal{X}$
3. (Bilinear) $\langle ax + bx', y \rangle = a \langle x, y \rangle + b \langle x', y \rangle, \forall a, b \in \mathbb{R} \in \ x, x', y \in \mathcal{X}$

Any inner product space $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ is a normed space (and hence in turn a metric space) since any inner product $\langle \cdot, \cdot \rangle$ defines the **induced norm** $\|x\| \doteq \sqrt{\langle x, x \rangle}$ such that $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ is a valid normed (vector) space called a **pre-Hilbert space** $(\mathcal{X}, \|\cdot\|)$. As such the definitions provided in Section 2.3 and Section 2.4 all apply to inner product spaces. A pre-Hilbert space that is complete is called a **Hilbert space**. Throughout this thesis, unless otherwise mentioned, we will be working with the ℓ_n^2 inner product spaces $(\mathbb{R}^n, \langle \cdot, \cdot \rangle)$ where $\langle \boldsymbol{x}, \boldsymbol{y} \rangle \doteq \sum_{i=1}^n x_i y_i$, whose Hilbert space is given by the ℓ_n^2 normed space which itself corresponds to the Euclidean metric space.

2.6 Measure and Probability Spaces

A **measurable space** $(\mathcal{X}, \mathcal{F})$ consists of a set $\mathcal{X}$ and a collection $\mathcal{F}$ of subsets of $\mathcal{X}$ which satisfies the following conditions:

1. (Closure under finite unions) for all $\mathcal{B}_1, \dots \mathcal{B}_n \in \mathcal{F}$, $\bigcup_{i=1}^n \mathcal{B}_i \in \mathcal{F}$
2. (Closure under complements) for all $\mathcal{B} \in \mathcal{F}$, $\mathcal{B}^c \in \mathcal{F}$,

Let $(\mathcal{X}, \mathcal{F}_\mathcal{X})$ and $(\mathcal{Y}, \mathcal{F}_\mathcal{Y})$ be two measurable spaces. A function $f : \mathcal{X} \to \mathcal{Y}$ is said to be a measurable iff for all $\mathcal{B} \in \mathcal{F}_\mathcal{Y}$, the inverse image of $\mathcal{B}$ is contained in $\mathcal{F}_\mathcal{X}$, i.e., $f^{-1}(\mathcal{B}) \in \mathcal{F}_\mathcal{X}$. If $f : \mathcal{X} \to \mathcal{Y}$ is a measurable function, we will often write $f : (\mathcal{X}, \mathcal{F}_\mathcal{X}) \to (\mathcal{Y}, \mathcal{F}_\mathcal{Y})$.

A **measure** $\mu : \mathcal{F} \to \mathbb{R}_+$ on a measurable space $(\mathcal{X}, \mathcal{F})$ is a function which satisfies the following conditions:

1. (Normalized) $\mu(\emptyset) = 0$
2. (Countable additivity) for all pairwise disjoint collection of sets $\{\mathcal{B}_i\}_{i=1}^\infty \subseteq \mathcal{F}$, $\mu(\bigcup_{i=1}^\infty \mathcal{B}_i) = \sum_{i=1}^\infty \mu(\mathcal{B}_i)$.

A **measure space** $(\mathcal{X}, \mathcal{F}, \mu)$ is a triple which consists of a measurable space $(\mathcal{X}, \mathcal{F})$ and a measure μ on $(\mathcal{X}, \mathcal{F})$. Let $\mathcal{X}$ be any set and $(\mathcal{X}, \mathcal{F})$ an associated measurable space, we define $\Delta(\mathcal{X}, \mathcal{F}) \doteq \{\mu : (\mathcal{X}, \mathcal{F}) \to [0, 1]\}$ denote the set of probability measures on $(\mathcal{X}, \mathcal{F})$. When $\mathcal{F}$ is clear from context, we simply write $\Delta(\mathcal{X})$. We also define the **support** of a measure $\mu \in \Delta(\mathcal{X})$ as $\text{supp}(\mu) \doteq \{x \in \mathcal{X} : \mu(x) > 0\}$.

A **simple function** $s : \mathcal{X} \to \mathbb{R}_+$ is a measurable function of the form $s(x) \doteq \sum_{i=1}^n \alpha_i \mathbb{1}_{\mathcal{Y}_i}(x)$ for some $\{\alpha_i\}_i \subseteq \mathbb{R}_+$ and $\{\mathcal{Y}_i\}_i \subseteq \mathcal{F}$. The (Lebesgue) integral of a simple function s over a set $\mathcal{B} \in \mathcal{F}$ is defined as:

$$\int_{x \in \mathcal{B}} s(x) d\mu(x) \doteq \sum_{i=1}^n \alpha_i \mu(\mathcal{Y}_i \cap \mathcal{B}) \tag{2.2}$$

The (Lebesgue) integral of a positive measurable function $f : \mathcal{X} \to \mathbb{R}_+$ is defined as:

$$\int_{x \in \mathcal{B}} f(x) d\mu(x) \doteq \sup \left\{ \int_{x \in \mathcal{B}} s(x) d\mu(x) \mid s \text{ is a simple function and } \forall x \in \mathcal{X}, s(x) \leq f(x) \right\} \tag{2.3}$$

We then extend this definition of the (Lebesgue) integral to any measurable function $f : \mathcal{X} \to \mathbb{R}$ by defining:

$$\int_{x \in \mathcal{B}} f(x) d\mu(x) \doteq \int_{x \in \mathcal{B}} \max\{f(x), 0\} \, d\mu(x) - \int_{x \in \mathcal{B}} \max\{-f(x), 0\} d\mu(x) \tag{2.4}$$

When clear from context, we will often denote $\int_{x \in \mathcal{B}} f(x) d\mu(x)$ by $\int_{\mathcal{B}} s d\mu$ or $\int_{\mathcal{B}} f(x) dx$.

A measurable function $f : \mathcal{X} \to \mathbb{R}$ is **integrable** on $\mathcal{B}$ iff $\int_{x \in \mathcal{B}} |f(x)| d\mu(x) < \infty$ where $x \mapsto |x|$ is the absolute function. We note that any bounded measurable function $f : \mathcal{X} \to \mathbb{R}$ is integrable on $\mathcal{B}$ if $\mu(\mathcal{B}) < \infty$. If f is integrable on $\mathcal{X}$, then f is said to be **integrable**.

A **probability space** is a measure space $(\mathcal{O}, \mathcal{E}, \mu)$ where

1. $\mathcal{O}$ is called the **sample space** and its elements are called **outcomes**
2. $\mathcal{E}$ is the event space which consists of sets of outcomes
3. $\mu : \mathcal{F} \to [0, 1]$ is a probability measure which satisfies $\mu(\mathcal{F}) = 1$

Given a measurable space $(\mathcal{X}, \mathcal{F})$ and a measure probability space $(\mathcal{O}, \mathcal{E}, \mu)$, a **random variable** is a measurable function $X : (\mathcal{O}, \mathcal{E}) \to (\mathcal{X}, \mathcal{F})$. A random variable maps outcomes to elements of a measurable allowing us to quantify the occurrence of random outcomes. Throughout this thesis, we will denote random variables with normal font capital letters.

The probability that a random variable X takes on a value $x \in \mathcal{X}$ is denoted as:

$$\mathbb{P}_{X \sim \mu}(X = x) \doteq \mu(\{o \in \mathcal{O} \mid X(o) = x\}) \tag{2.5}$$

Similarly, probability that a random variable X takes on a value in the set $\mathcal{Y} \subseteq \mathcal{X}$ is denoted as:

$$\mathbb{P}_{X \sim \mu}(X \in \mathcal{Y}) \doteq \mu(\{o \in \mathcal{O} \mid X(o) \in \mathcal{Y}\}) \tag{2.6}$$

The **expectation** of a random variable X is defined as:

$$\mathop{\mathbb{E}}_{X \sim \mu}[X] \doteq \int_{\mathcal{O}} X d\mu \tag{2.7}$$

2.7 (Sub)differential Calculus

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be **normed spaces**. Consider an operator $f : (\mathcal{X}, \|\cdot\|_{\mathcal{X}}) \to (\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$. A linear operator $f : \mathcal{X} \to \mathcal{Y}$ is said to be **bounded** if there exists $c < \infty$ such that for all $x \in \mathcal{X}$:

$$\|f(x)\|_{\mathcal{X}} \leq c\|x\|_{\mathcal{Y}} \tag{2.8}$$

The **dual (vector) space** $\mathcal{X}^*$ of any (vector) space $\mathcal{X}$ consists of all linear functions $f : \mathcal{X} \to \mathbb{R}$ and is associated with a **dual normed vector space** $(\mathcal{X}^*, \|\cdot\|^*_{\mathcal{X}})$ where $\|f\|^*_{\mathcal{X}} \doteq \sup_{\substack{x \in \mathcal{X} \\ \|x\| \leq 1}} \|f(x)\|_{\mathcal{X}}$.

The **directional (or Gâteau) derivative** (Gâteaux, 1913) of an operator $f : \mathcal{X} \to \mathcal{Y}$ at $\widehat{x} \in \mathcal{X}$ in the direction of $a \in \mathcal{X}$ is a linear function $\nabla_{\widehat{x}} f(a) \in \mathcal{X}^*$ on $\mathcal{X}$ s.t.:

$$\nabla_a f(\widehat{x}) = \lim_{t \to 0} \frac{f(\widehat{x} + ta) - f(\widehat{x})}{t} \tag{2.9}$$

f is said to be **differentiable**, if for all $\widehat{x}, a \in \mathcal{X}$, $\nabla_{\boldsymbol{a}} f(\widehat{x})$ exists. f is said to be **continuously differentiable** if for all $a \in \mathcal{X}$, $x \mapsto \nabla_{\boldsymbol{a}} f(x)$ is continuous. If $\mathcal{X} \subseteq \mathbb{R}^n$ and $\mathcal{Y} \doteq \mathbb{R}$, overloading notation, we define the **partial derivative** of the function $f : \mathcal{X} \to \mathbb{R}$ w.r.t x_i for all $i \in [n]$ at $\widehat{\boldsymbol{x}} \in \mathcal{X}$ as $\nabla_{x_i} f(\widehat{\boldsymbol{x}}) \doteq \nabla_{\boldsymbol{j}_i} f(\widehat{\boldsymbol{x}})$, and the gradient (or Fréchet derivative) of $f : \mathcal{X} \to \mathbb{R}$ at $\widehat{\boldsymbol{x}} \in \mathcal{X}$ as $\nabla f(\widehat{\boldsymbol{x}}) = \nabla_{\boldsymbol{x}} f(\widehat{\boldsymbol{x}}) \doteq (\nabla_{x_i} f(\widehat{\boldsymbol{x}}))_{i=1}^n$.

The **(Clarke) subdifferential** (Clarke, 1990) of a function $f : \mathcal{X} \to \mathcal{Y}$, is a set $\mathcal{D}f(x) \subseteq \mathcal{X}^*$ of linear functions on $\mathcal{X}$ defined as $\mathcal{D}f(x) \doteq \operatorname{conv}\left\{\lim_{k \to \infty} \nabla f(x^{(k)}) \mid \exists x^{(k)} \to x \text{ s.t. } x^{(k)} \in \operatorname{dom}(\nabla f))\right\}$. Analogously, we define the **directional (Clarke) subdifferential** $\mathcal{D}_x$ w.r.t x by replacing the gradient operator in the definition by the directional derivative, i.e., ∇_x. To simplify notation, we often write $\partial_x f(\widehat{x})$ to refer to an arbitrary **subgradient** (i.e., an element of the subdifferential) of f at x, e.g., $\partial_x f(\widehat{x}) \in \mathcal{D}_x f(\widehat{x})$.

When f is continuously differentiable, by the definition of continuity, the subdifferential is singleton-valued and for all $\widehat{x} \in \mathcal{X}$, $\mathcal{D}_x f(\widehat{x}) \doteq \{\nabla_x f(\widehat{x})\}$. A function f is said to be **subdifferentiable** iff its subdifferential is non-empty for all points in its domain, i.e., for all $x \in \mathcal{X}$, $\partial_x f(x) \neq \emptyset$. We note that a function is subdifferentiable if it is locally-Lipschitz continuous[6] (Clarke, 1990). Throughout this thesis we will work only with subdifferentiable functions, and as such for any function $f : \mathcal{X} \to \mathcal{Y}$ we will define its **subdifferential correspondence** $\mathcal{D}f : \mathcal{X} \rightrightarrows \mathcal{Y}$ as the correspondence which takes as input a point in the

[6]We refer the reader to Section 2.8 for a definition.

domain $x \in \mathcal{X}$, and outputs the subdifferential $\mathcal{D}f(x)$ of f at x. Similarly, we also define the directional subdifferential correspondence $\mathcal{D}_x f : \mathcal{X} \rightrightarrows \mathcal{Y}$ of any subdifferentiable function f. We note that for any function f, the subdifferential is $\mathcal{D}f$ is upper hemicontinuous, non-empty-, and compact-valued (Clarke, 2007). For any continuous and convex function $f : \mathcal{X} \to \mathbb{R}$, the subdifferential correspondence $\mathcal{D}f$ is upper hemicontinuous non-empty-, compact-, and convex-valued (see Theorem 24.4 of Pryce (1973)).

2.8 Primitive Function Structures

In this section, we introduce the key definitions that will be used to derive the results in this thesis. These are refinements of the notions of continuity and convexity for functions, and monotonicity correspondences.

Consider a complete order $(\mathcal{U}, \mathcal{R})$, and a function $f : \mathcal{U} \to \mathbb{R}$. f is said to be **increasing** (resp. **decreasing**) over $\mathcal{X} \subseteq \mathcal{U}$ iff for all $x, y \in \mathcal{X}$ s.t. $x \succeq_{\mathcal{R}} y$, $f(x) \geq f(y)$ (resp. $f(x) \leq f(y)$). f is said to be **strictly increasing** (resp. **strictly decreasing**) iff for all $x, y \in \mathcal{X}$ s.t. $x \succ_{\mathcal{R}} y$, $f(x) > f(y)$ (resp. $f(x) < f(y)$).

Monotonicity Properties of Correspondences

To obtain our computational results, we will rely on generalized monotonicity properties of correspondences. Let $(\mathcal{X}, \langle \cdot, \cdot \rangle)$ be an inner product space. Consider a correspondence $\mathcal{R} : \mathcal{X} \rightrightarrows \mathcal{X}$.

Definition 2.8.1 [Weakly-Monotone/Dissipative Correspondences]**.**
$\mathcal{R}$ is μ**-weakly-monotone** with **modulo of monotonocity** $\mu \in \mathbb{R}$ iff

$$\left\langle y' - y, x' - x \right\rangle \geq -\mu \|x - x'\|^2 \qquad \forall y \in \mathcal{R}(x), y' \in \mathcal{R}(x')$$

$\mathcal{R}$ is ν**-weakly-dissipative** with **modulo of dissipativity** $\nu \in \mathbb{R}$ iff

$$\left\langle y' - y, x' - x \right\rangle \leq \nu \|x - x'\|^2 \qquad \forall y \in \mathcal{R}(x), y' \in \mathcal{R}(x')$$

Note that in the above definition when $\mu < 0$ (resp. $\nu < 0$), a μ-weakly-monotone (resp. ν-weakly-dissipative) correspondence is often called $(-\mu)$**-strongly-monotone** (resp. $(-\nu)$**-strongly-dissipative**.

In the special case that $\mu \doteq 0$, and $\nu \doteq 0$, we recover the definition of monotone and dissipative operators:

Definition 2.8.2 [Monotone/Dissipative Correspondences]**.**
$\mathcal{R}$ is **monotone** iff:

$$\left\langle y' - y, x' - x \right\rangle \geq 0 \qquad \forall y \in \mathcal{R}(x), y' \in \mathcal{R}(x')$$

$\mathcal{R}$ is **dissipative** iff:

$$\left\langle y' - y, x' - x \right\rangle \leq 0 \qquad \forall y \in \mathcal{R}(x), y' \in \mathcal{R}(x')$$

Definition 2.8.3 [Pseudomonotone/Pseudodissipative]**.**

$\mathcal{R}$ is **pseudomonotone** iff:

$$\left\langle y', x' - x \right\rangle \geq 0 \implies \left\langle y, x - x' \right\rangle \geq 0 \qquad \forall y \in \mathcal{R}(x), y' \in \mathcal{R}(x')$$

$\mathcal{R}$ is **pseudodissipative** iff:

$$\left\langle y', x' - x \right\rangle \leq 0 \implies \left\langle y, x - x' \right\rangle \leq 0 \qquad \forall y \in \mathcal{R}(x), y' \in \mathcal{R}(x')$$

Definition 2.8.4 [quasimonotone/quasidissipative]**.**

Consider a correspondence $\mathcal{R} : \mathcal{X} \rightrightarrows \mathcal{X}$.

$\mathcal{R}$ is **quasimonotone** iff:

$$\left\langle y', x' - x \right\rangle > 0 \implies \left\langle y, x - x' \right\rangle \geq 0 \qquad \forall y \in \mathcal{R}(x), y' \in \mathcal{R}(x')$$

$\mathcal{R}$ is **quasidissipative** iff:

$$\left\langle y', x' - x \right\rangle < 0 \implies \left\langle y, x - x' \right\rangle \leq 0 \qquad \forall y \in \mathcal{R}(x), y' \in \mathcal{R}(x')$$

We note the following relationships between the above properties:

$$\text{monotone} \implies \text{pseudomonotone} \implies \text{quasimonotone} \tag{2.10}$$

$$\text{dissipative} \implies \text{pseudodissipative} \implies \text{quasidissipative} \tag{2.11}$$

Lipschitz Properties of Functions

Let $(\mathcal{X}, \|\cdot\|_{\mathcal{X}})$ and $(\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$ be normed spaces. Consider a function $f : (\mathcal{X}, \|\cdot\|_{\mathcal{X}}) \to (\mathcal{Y}, \|\cdot\|_{\mathcal{Y}})$. We will first consider refinements of the notions of continuity and continuous differentiability which we will us to derive the complexity results in this thesis.

Definition 2.8.5 [Lipschitz-Continuity]**.**

A function $f : \mathcal{X} \to \mathcal{Y}$ is said to be ℓ_f-**Lipschitz-continuous on** $\mathcal{S} \subseteq \mathcal{X}$ iff for all $x_1, x_2 \in \mathcal{X}$

$$\|f(x_1) - f(x_2)\|_{\mathcal{Y}} \leq \ell_f \|x_1 - x_2\|_{\mathcal{X}} , \tag{2.12}$$

When $\mathcal{S} = \mathcal{X}$, then f is simply said to be ℓ_f-**Lipschitz-continuous**.

We note that any continuously differentiable function f on a non-empty, and compact set $\mathcal{S}$ is guaranteed to be ℓ-Lipschitz-continuous on $\mathcal{S}$ with $\ell \doteq \max_{x \in \mathcal{S}} \|\nabla f(x)\|_{\mathcal{X}^*}$.

An important generalization of Lipschitz continuity is local-Lipschitz-continuity, a class of functions which are (Clarke) subdifferentiable.

Definition 2.8.6 [Local-Lipschitz-Continuity]**.**
A function $f : \mathcal{X} \to \mathcal{Y}$ is said to be ℓ_f-**locally Lipschitz-continuous** iff there exists $\varepsilon > 0$ s.t. for all $x \in \mathcal{X}$, f is ℓ_f-Lipschitz continuous on $\mathcal{B}_\varepsilon(x)$.

We note that any locally Lipschitz continuous function is continuous but not vice-versa.

An important refinement of continuous continuous differentiability which has been used pervasively by prior work (see, for instance, Daskalakis et al. (2020b)) is the notion of Lipschitz-smoothness, which requires the gradient of a function to Lipschitz-smooth.

Definition 2.8.7 [Lipschitz-Smoothness]**.**
A function $f : \mathcal{X} \to \mathcal{Y}$ is said to be λ**-Lipschitz-smooth on** $\mathcal{S} \subseteq \mathcal{X}$ iff for all $x_1, x_2 \in \mathcal{X}$

$$\|\nabla f(x_1) - \nabla f(x_2)\|_{\mathcal{Y}} \leq \lambda \|x_1 - x_2\|_{\mathcal{X}} , \tag{2.13}$$

When $\mathcal{S} = \mathcal{X}$, then f is simply said to be λ-**Lipschitz-smooth**.

Convexity Properties of Functions

Definition 2.8.8 [Convex/Concave Function]**.**
Consider a function $f : \mathcal{X} \to \mathbb{R}$.

f is **convex** iff for all $\lambda \in [0, 1]$ and $x, x' \in \mathcal{X}$,

$$f(\lambda x + (1-\lambda)x') \leq \lambda f(x) + (1-\lambda) f(x') \ .$$

f is **concave** iff for all $\lambda \in [0, 1]$ and $x, x' \in \mathcal{X}$,

$$f(\lambda x + (1-\lambda)x') \geq \lambda f(x) + (1-\lambda) f(x') \ .$$

f is said to be **affine** iff it is both convex and concave.

Note that, a function f is convex iff $-f$ is concave.

When f is convex, its subdifferential correspondence $\mathcal{D}f$ is called the **convex subdifferential** and is given as $\mathcal{D}f(x) \doteq \{x^* \in \mathcal{X}^* \mid f(x') \geq f(x) + \langle x^*, x' - x \rangle, \forall x' \in \mathcal{X}\}$ (see Theorem 25.6 of Ralph Tyrell (1997)). Similarly, when f is concave its Clarke subdifferential correspondence $\mathcal{D}f$ is called the **concave subdifferential** and is given as $\mathcal{D}f(x) \doteq \{x^* \in \mathcal{X}^* \mid f(x') \leq f(x) + \langle x^*, x' - x \rangle, \forall x' \in \mathcal{X}\}$.

Definition 2.8.9 [Quasiconvex/Quasiconcave Functions]**.**

Consider a function $f : \mathcal{X} \to \mathbb{R}$.

f is **quasiconvex** iff for all $\lambda \in (0, 1)$ and $x, x' \in \mathcal{X}$,

$$f(\lambda x + (1 - \lambda)x') \leq \max\{f(x), f(x')\} \quad . \tag{2.14}$$

f is **quasiconcave** iff for all $\lambda \in (0, 1)$ and $x, x' \in \mathcal{X}$,

$$f(\lambda x + (1 - \lambda)x') \geq \min\{f(x), f(x')\} \quad . \tag{2.15}$$

In addition, a function f is quasiconvex iff $-f$ is quasiconcave. We note that a function f is quasiconvex iff its **sublevel sets**, i.e., the sets $\{x \in \mathcal{X} \mid f(x) \leq \alpha\}$ for all $\alpha \in \mathbb{R}$, is convex. Similarly, a function f is quasiconcave iff its **superlevel set**, i.e., the set $\{x \in \mathcal{X} \mid f(x) \geq \alpha\}$ for all $\alpha \in \mathbb{R}$, is convex. Quasiconvex and quasiconcave functions are very useful in representing a class of continuous and convex sets or correspondences. In particular, consider metric spaces $(\mathcal{X}, d_{\mathcal{X}})$, $(\Theta, d_{\mathcal{X}})$, and some continuous quasiconvex functions $g_1, \ldots, g_l : \mathcal{X} \times \mathcal{Y}$, then the correspondence $\mathcal{R}(y) \doteq \{x \in \mathcal{X} \mid g_i(x, y) \leq 0, i \in [l]\}$ is continuous and convex (see Theorem 5.9 of Rockafellar and Wets (2009)).

Going beyond classes of convex/concave functions, this thesis will make use of notions of weakly-convex/concave functions. The class of weakly-convex/concave functions were first introduced to the optimization literature in English by Nurminskii (1973), and have become a class of functions of great interest in the optimization literature is the recent years (see, for instance, (Davis et al., 2018; Davis and Drusvyatskiy, 2019; Lin et al., 2020)).

Definition 2.8.10 [Weakly-Convex/Weakly-Concave Functions]**.**

Consider a function $f : \mathcal{X} \to \mathbb{R}$.

f is μ-**weakly-convex** with **modulus of convexity** $\mu \in \mathbb{R}$, iff $x \mapsto f(x) + \frac{\mu}{2}\|x\|^2$ is convex. If $\mu < 0$, then f is said to be $(-\mu)$-**strongly-convex**.[7]

f is ν-**weakly-concave** with **modulus of concavity** $\nu \in \mathbb{R}$, iff $x \mapsto f(x) - \frac{\mu}{2}\|x\|^2$ is concave. If $\mu < 0$, then f is said to be $(-\mu)$-**strongly-convex**.

Remark 2.8.1 [Examples of Weakly-Convex/Concave Functions]**.**
Naturally, the class of weakly-convex (resp. weakly-concave) functions generalizes strongly-convex (resp. weakly-concave) functions, and convex/concave functions, and convex functions.

More importantly, however, any ℓ-smooth function is both ℓ-weakly-convex and ℓ-weakly-concave Davis and Drusvyatskiy (2019). In some sense, Lipschitz-smooth functions can be interpreted as being weakly-affine (i.e., both weakly-convex and weakly-concave). Yet, despite Lipschitz-smooth functions being a very restricted subset of the class of weakly-convex (resp. weakly-concave) functions, they contain a very large class of non-convex and differentiable functions. In fact, the class of Lipschitz-smooth functions on a non-empty, and compact domain contains amongst others all twice continuously differentiable functions. We refer the reader to Section 2.1 of Davis and Drusvyatskiy (2019) and Section 4 of Vial (1983) for additional results and discussions on weak-convexity and weak-concavity.

Remark 2.8.2 [Subdifferential of Weakly-Convex/Concave Functions]**.**
Any weakly-convex (resp. weakly-concave) function f is subdifferentiable as it can be re-written as the difference of two convex functions ($f + \mu/2\|\cdot\|^2$ and $\mu/2\|\cdot\|^2$ (resp. concave functions ($f - \mu/2\|\cdot\|^2$ and $\mu/2\|\cdot\|^2$); as convex (resp. concave) functions are locally Lipschitz continuous so is their difference, which in turn implies subdifferentiability by Theorem 3.1. of Clarke (2007).

In addition, for any μ-weakly-convex (resp. ν-weakly-concave) function f, its subdifferential correspondence is given as

$$\mathcal{D}f(x) \doteq \mathcal{D}[f(x) + \tfrac{\mu}{2}\|x\|^2] - \mu x \quad \left(\text{resp. } \mathcal{D}f(x) = \mathcal{D}[f(x) - \tfrac{\nu}{2}\|x\|^2] + \mu x\right) \ ,$$

where $\mathcal{D}[f(x) + \frac{\mu}{2}\|x\|^2]$ (resp. $\mathcal{D}[f(x) - \frac{\nu}{2}\|x\|^2]$ is the convex (resp. concave) subdifferential, since $f(\cdot) + \frac{\mu}{2}\|\cdot\|^2$ is convex (resp. $\mathcal{D}[f(x) - \frac{\nu}{2}\|x\|^2]$ is concave).

[7] See, for instance, Section 9.1.2 of Boyan and Moore (1994) for further characterizations.

An implication of this remark is that, as a subgradient of a convex/concave function can be computed (or in the worst-case approximated) easily (see, for instance Bertsekas (2011))via convex/concave subdifferential calculus rules, a subgradient of a weakly-convex/weakly-concave function can also be computed easily. As such, as is standard in the literature (see for instance Lin et al. (2020)), we will take the number of subgradient evaluations as the primitive operation of our computational complexity results.

In addition, for weakly-convex and weakly-concave functions, we have the following characterization introduced by Davis and Drusvyatskiy (see Lemma 2.1 of Davis and Drusvyatskiy (2019)) whose proof we provide for completness. Note that as a function f is μ-weakly-convex iff $-f$ is μ-weakly-concave, an analogous characterization holds for weakly-concave functions as well.

Lemma 2.8.1 [Characterization of Weakly-Convex-Functions]**.**
Consider a function $f : (\mathcal{X}, \|\cdot\|) \to \mathbb{R}$, and a modulus of convexity $\mu \in \mathbb{R}$. The following statements are equivalent:

1. f is μ-weakly-convex
2. The μ-**weak secant inequality** holds:
$$f(\lambda x + (1-\lambda)x') \leq \lambda f(x) + (1-\lambda) f(x') + \frac{\mu\lambda(1-\lambda)}{2}\|x - x'\|^2 \tag{2.16}$$
3. The μ-**weak subgradient inequality** holds, i.e.,
$$f(x') \geq f(x) + \left\langle \partial f(x), x' - x \right\rangle - \mu/2\|x - x'\|^2, \qquad \forall x, x' \in \mathcal{X}, \partial f(x) \in \mathcal{D}f(x)$$
4. The subdifferential map is μ-**weakly-monotone**, i.e.,
$$\left\langle \partial f(x') - \partial f(x), x' - x \right\rangle \geq -\mu\|x - x'\|^2 \qquad \forall \partial f(x) \in \mathcal{D}f(x), \partial f(x') \in \mathcal{D}f(x')$$

Proof of Lemma 2.8.1

Fix $\mu \in \mathbb{R}$, and let $f : \mathcal{X} \to \mathbb{R}$ be a μ-weakly-convex function. We then have:

(1) $\equiv$ (2):

From the definition of weak-convexity, we have for all $x, x' \in \mathcal{X}$ and $\lambda \in [0,1]$:

$$f(\lambda x + (1-\lambda)x') + {}^{\mu}\!/_{2}\|\lambda x + (1-\lambda)x'\|^2 \leq \lambda[f(x) + {}^{\mu}\!/_{2}\|x\|^2] + (1-\lambda)[f(x') + {}^{\mu}\!/_{2}\|x'\|^2]$$
$$\leq \lambda f(x) + (1-\lambda)f(x') + {}^{\lambda\mu}\!/_{2}\|x\|^2 + {}^{(1-\lambda)\mu}\!/_{2}\|x'\|^2 \ .$$

Re-organizing the expression, we get:

$$f(\lambda x + (1-\lambda)x') \leq \lambda f(x) + (1-\lambda)f(x') + {}^{\lambda\mu}\!/_{2}\|x\|^2 + {}^{(1-\lambda)\mu}\!/_{2}\|x'\|^2 - {}^{\mu}\!/_{2}\|\lambda x + (1-\lambda)x'\|^2 \tag{2.17}$$

Now, notice that we have for all $x, x' \in \mathcal{X}$ and $\lambda \in [0,1]$:

$$\begin{aligned}
&{}^{\lambda\mu}\!/_{2}\|x\|^2 + {}^{(1-\lambda)\mu}\!/_{2}\|x'\|^2 - {}^{\mu}\!/_{2}\|\lambda x + (1-\lambda)x'\|^2 \\
&= {}^{\lambda\mu}\!/_{2}\|x\|^2 + {}^{(1-\lambda)\mu}\!/_{2}\|x'\|^2 - {}^{\mu\lambda^2}\!/_{2}\|x\|^2 - \lambda(1-\lambda)\mu\left\langle x, x'\right\rangle - {}^{(1-\lambda)^2\mu}\!/_{2}\|x'\|^2 \\
&= {}^{\lambda\mu}\!/_{2}\|x\|^2 - {}^{\mu\lambda^2}\!/_{2}\|x\|^2 + {}^{(1-\lambda)\mu}\!/_{2}\|x'\|^2 - {}^{(1-\lambda)^2\mu}\!/_{2}\|x'\|^2 - \lambda(1-\lambda)\mu\left\langle x, x'\right\rangle \\
&= {}^{\lambda(1-\lambda)\mu}\!/_{2}\|x\|^2 + {}^{\lambda(1-\lambda)\mu}\!/_{2}\|x'\|^2 - \lambda(1-\lambda)\mu\left\langle x, x'\right\rangle \\
&= \frac{\lambda(1-\lambda)\mu}{2}\|x - x'\|^2
\end{aligned}$$

Hence, plugging the above to Equation (2.17), we get for all $x, x' \in \mathcal{X}$ and $\lambda \in [0,1]$:

$$f(\lambda x + (1-\lambda)x') \leq \lambda f(x) + (1-\lambda)f(x') + \frac{\lambda(1-\lambda)\mu}{2}\|x - x'\|^2$$

As all the inequalities are tight, the implication holds both ways.

$(2) \equiv (3)$

Re-organizing the terms in Equation (2.16), we have for all $x, x' \in \mathcal{X}$ and $\lambda \in [0,1]$:

$$f(\lambda x + (1-\lambda)x') \leq \lambda f(x) + (1-\lambda)f(x') + {}^{\lambda(1-\lambda)\mu}\!/_{2}\|x - x'\|^2$$
$$f(x' + \lambda(x - x')) \leq \lambda f(x) + (1-\lambda)f(x') + {}^{\lambda(1-\lambda)\mu}\!/_{2}\|x - x'\|^2$$
$$f(x' + \lambda(x - x')) - f(x') \leq \lambda\left[f(x) - f(x')\right] + {}^{\lambda(1-\lambda)\mu}\!/_{2}\|x - x'\|^2$$

Dividing both sides of the inequality by λ, for all $x, x' \in \mathcal{X}$ and $\lambda \in [0,1]$ we have:

$$\frac{f(x' + \lambda(x - x')) - f(x)}{\lambda} \leq f(x) - f(x') + {}^{(1-\lambda)\mu}\!/_{2}\|x - x'\|^2$$

Now, taking the limit as $\lambda \to 0$, we have for all $x, x' \in \mathcal{X}$ the μ-subgradient inequality for all possible subgradients:

$$\lim_{\lambda \to 0} \frac{f(x' + \lambda(x' - x)) - f(x')}{\lambda} \leq f(x) - f(x') + {}^{(1-\lambda)\mu}\!/_{2}\|x - x'\|^2$$

Note that left hand side is the definition of the Gâteu derivative, hence by varying $x, \in \mathcal{X}$ we obtain all possible limits points of the Gâteu derivative, as this set of limit points is closed and convex, then it is equal to its convex hull, which is the definition of the Clarke subdifferential. In addition, once again, as we have applied no inequalities, the bounds are tight and the implication holds both ways.

$(3) \equiv (4)$:

By the μ-subgradient inequality, we have the two following relations:

$$f(x') \geq f(x) + \left\langle \partial f(x), x' - x \right\rangle - {}^{\nu}\!/\!{}_{2} \|x - x'\|^2, \qquad \forall x, x' \in \mathcal{X}, \partial f(x) \in \mathcal{D} f(x)$$

$$f(x) \geq f(x') + \left\langle \partial f(x'), x - x' \right\rangle - {}^{\nu}\!/\!{}_{2} \|x - x'\|^2, \qquad \forall x, x' \in \mathcal{X}, \partial f(x') \in \mathcal{D} f(x')$$

Subtracting the first inequality from the second, and re-organizing the terma, we then obtain the μ-weak-monotonicity condition. The reverse direction follows in the same way.

2.9 Constrained Optimization Background

2.9.1 The Primal Problem

Consider any Euclidean metric space $(\mathbb{R}^n, d)$. A **(constrained) optimization problem** $\mathcal{C} \doteq (n, l, \mathcal{X}, f, \boldsymbol{g})$, denoted $(\mathcal{X}, f, \boldsymbol{g})$ when clear from context, consists of an **objective function** $f : \mathbb{R}^n \to \mathbb{R}$, $l \in \mathbb{N}$ **constraint functions** $g_1, \ldots, g_l : \mathbb{R}^n \to \mathbb{R}$, and a basic feasible set $\mathcal{X} \subseteq \mathbb{R}^n$ which define the following maximization problem called the **Primal Problem**[8]:

Primal Problem

$$\max_{\boldsymbol{x} \in \mathcal{X}} \quad f(\boldsymbol{x}) \tag{2.18}$$

$$\text{Constrained by} \quad g_i(\boldsymbol{x}) \geq 0 \qquad \forall i \in [l] \tag{2.19}$$

For convenience, we will denote $\boldsymbol{g} \doteq (g_i)_{i=1}^{l} : \mathbb{R}^n \to \mathbb{R}^{l+p}$, and in consistence with the partial order we have defined on Euclidean vector spaces, we will often write $\boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0}$ to mean for all $i \in [l]$, $g_i(\boldsymbol{x}) \geq 0$. We define the **feasible set** $\operatorname{feas}(\mathcal{X}, f, \boldsymbol{g})$ of the optimization problem $(\mathcal{X}, f, \boldsymbol{g})$ as:

$$\operatorname{feas}(\mathcal{X}, f, \boldsymbol{g}) \doteq \{\boldsymbol{x} \in \mathcal{X} \mid \boldsymbol{g}(\boldsymbol{x}) \geq \boldsymbol{0}\}$$

A point $\boldsymbol{x} \in \mathbb{R}^n$ is said to be **feasible** if it is an element of the feasible set $\operatorname{feas}(\mathcal{X}, f, \boldsymbol{g})$. Notice that the primal problem can be restated as $\max_{\boldsymbol{x} \in \operatorname{feas}(\mathcal{X}, f, \boldsymbol{g})} f(\boldsymbol{x})$. As such, by the Extreme Value Theorem, under the following assumption which we will assume throughout this section, a solution to the primal problem exists:

Assumption 2.9.1 [Existence of Solution]**.**
Consider an optimization problem $(\mathcal{X}, f, \boldsymbol{g})$. Suppose that:

1. $f : \mathcal{X} \to \mathbb{R}$ is continuous
2. $\operatorname{feas}(\mathcal{X}, f, \boldsymbol{g})$ is non-empty and compact.

We note that Part 2 of Assumption 2.9.1, can be guaranteed under the assumption that $\boldsymbol{g}$ is continuous, and the feasible set $\operatorname{feas}(\mathcal{X}, f, \boldsymbol{g})$ non-empty.

[8] We will for convenience, without loss of generality, focus on maximization problems, since any maximization problem can be recast as a minimization by negating the objective functions.

2.9.2 The Lagrangian and the Dual Problem

For any optimization problem $(\mathcal{X}, f, \boldsymbol{g})$, we define the **Lagrangian (function),** $\ell : \mathbb{R}^n \times \mathbb{R}^l_+ \to \mathbb{R}$:

$$\ell(\boldsymbol{x}, \boldsymbol{\lambda}) \doteq f(\boldsymbol{x}) + \sum_{i=1}^{l} \lambda_i g_i(\boldsymbol{x})$$

where $\boldsymbol{\lambda} \doteq (\lambda_1, \dots, \lambda_l) \in \mathbb{R}^l_+$ are called **slack variables** (or **KKT multipliers**). These variables are called slack variables as they relax the constrained problem to an unconstrained one and by selecting them wisely allow us to obtain a function whose maximum over $\mathcal{X}$ corresponds exactly to those of the primal problem. More formally, take the infimum of Lagrangian over the respective domains of the slack variables, we have:

$$\begin{aligned}\inf_{\boldsymbol{\lambda} \geq \boldsymbol{0}} \ell(\boldsymbol{x}, \boldsymbol{\lambda}) &= \inf_{\boldsymbol{\lambda} \geq \boldsymbol{0}} \left[f(\boldsymbol{x}) + \sum_{i=1}^{l} \lambda_i g_i(\boldsymbol{x}) \right] \\ &= f(\boldsymbol{x}) + \inf_{\boldsymbol{\lambda} \geq \boldsymbol{0}} \sum_{i=1}^{l} \lambda_i g_i(\boldsymbol{x}) \\ &= f(\boldsymbol{x}) + \sum_{i=1}^{l} \inf_{\lambda_i \geq 0} \lambda_i g_i(\boldsymbol{x}) \end{aligned} \tag{2.20}$$

Notice that for all $i \in [l]$, we have the following:

$$\inf_{\lambda_i \geq 0} \lambda_i g_i(\boldsymbol{x}) = \begin{cases} 0 & \text{if } g_i(\boldsymbol{x}) \geq 0 \\ \infty & \text{Otherwise} \end{cases}$$

Going back to Equation (2.20), we then have:

$$\inf_{\boldsymbol{\lambda} \geq \boldsymbol{0}} \ell(\boldsymbol{x}, \boldsymbol{\lambda}) = \begin{cases} f(\boldsymbol{x}) & \text{if } \boldsymbol{g}(\boldsymbol{x}) \leq \boldsymbol{0} \\ \infty & \text{Otherwise} \end{cases}$$

That is, by taking the infimum of the Lagrangian over the slack variables $\boldsymbol{\lambda}$, we obtain a function where for all feasible points $\boldsymbol{x} \in \text{feas}(\mathcal{X}, f, \boldsymbol{g})$, the value of function coincides with the value of the objective f, and for all infeasible points $\boldsymbol{x}' \notin \text{feas}(\mathcal{X}, f, \boldsymbol{g})$ the value of the function is ∞. As a result, we have:

$$\max_{\boldsymbol{x} \in \text{feas}(\mathcal{X}, f, \boldsymbol{g})} f(\boldsymbol{x}) = \max_{\boldsymbol{x} \in \mathcal{X}} \inf_{\boldsymbol{\lambda} \geq \boldsymbol{0}} \ell(\boldsymbol{x}, \boldsymbol{\lambda})$$

The above equality suggests that if we could switch the order of the min and max on the right hand-side, we can re-express primal problem, as a minimization problem called the **dual problem**. To this end, define **The Lagrangian dual function** $\ell^* : \mathbb{R}^l_+ \to \bar{\mathbb{R}}$:

$$\ell^*(\boldsymbol{\lambda}) = \sup_{\boldsymbol{x} \in \mathcal{X}} \ell(\boldsymbol{x}, \boldsymbol{\lambda}) \tag{2.21}$$

The **dual problem** associated with any optimization problem $(\mathcal{X}, f, \boldsymbol{g})$ is then defined as:

$$\inf_{\boldsymbol{\lambda}} \quad \ell^*(\boldsymbol{\lambda}) \tag{2.22}$$

$$\text{Constrained by} \quad \boldsymbol{\lambda} \geq \boldsymbol{0} \tag{2.23}$$

A tuple of slack variable $\boldsymbol{\lambda} \in \mathbb{R}^l$ are said to be **feasible** iff $\boldsymbol{\lambda} \geq \boldsymbol{0}$.

We say that **weak duality holds** iff $\max_{\boldsymbol{x} \in \mathcal{X}} \inf_{\boldsymbol{\lambda} \geq \boldsymbol{0}} \ell(\boldsymbol{x}, \boldsymbol{\lambda}) \leq \inf_{\boldsymbol{\lambda} \geq \boldsymbol{0}} \sup_{\boldsymbol{x} \in \mathcal{X}} \ell(\boldsymbol{x}, \boldsymbol{\lambda})$. We say that **strong duality holds** iff $\max_{\boldsymbol{x} \in \mathcal{X}} \min_{\boldsymbol{\lambda} \geq \boldsymbol{0}} \ell(\boldsymbol{x}, \boldsymbol{\lambda}) = \min_{\boldsymbol{\lambda} \geq \boldsymbol{0}} \max_{\boldsymbol{x} \in \mathcal{X}} \ell(\boldsymbol{x}, \boldsymbol{\lambda})$. Note that for strong duality the infimum and supremum is replaced by a minimum and maximum since under Assumption 2.9.1 $\min_{\boldsymbol{x} \in \mathcal{X}} \sup_{\boldsymbol{\lambda} \geq \boldsymbol{0}} \ell(\boldsymbol{x}, \boldsymbol{\lambda})$ is well-defined and as such for equality to hold the supremum and infimum must be well-defined. Without the need for any additional assumptions, we can show that weak programming duality holds for any optimization problem.

Theorem 2.9.1 [Weak Duality]**.**
Consider an optimization problem $(\mathcal{X}, f, \boldsymbol{g})$ and suppose that Assumption 2.9.1 is satisfied, then weak duality holds.

Proof

$$\inf_{\boldsymbol{\lambda} \geq \boldsymbol{0}} \ell(\boldsymbol{x}, \boldsymbol{\lambda}) \leq \ell(\boldsymbol{x}, \boldsymbol{\lambda}') \qquad \forall \boldsymbol{x} \in \mathcal{X}, \boldsymbol{\lambda}' \geq \boldsymbol{0}$$

$$\sup_{\boldsymbol{x} \in \mathcal{X}} \inf_{\boldsymbol{\lambda} \geq \boldsymbol{0}} \ell(\boldsymbol{x}, \boldsymbol{\lambda}) \leq \sup_{\boldsymbol{x} \in \mathcal{X}} \ell(\boldsymbol{x}, \boldsymbol{\lambda}') \qquad \forall \boldsymbol{\lambda}' \geq \boldsymbol{0}$$

$$\sup_{\boldsymbol{x} \in \mathcal{X}} \inf_{\boldsymbol{\lambda} \geq \boldsymbol{0}} \ell(\boldsymbol{x}, \boldsymbol{\lambda}) \leq \inf_{\boldsymbol{\lambda} \geq \boldsymbol{0}} \sup_{\boldsymbol{x} \in \mathcal{X}} \ell(\boldsymbol{x}, \boldsymbol{\lambda})$$

In contrast, to ensure that strong duality holds, we have to considerably restrict class of optimization problems we consider, namely to regular convex optimization problems (i.e., optimization problems where f and $\boldsymbol{g}$ are convex, and for which a constraint qualification such as Slater's condition is satisfied), by making the following additional assumption:

Assumption 2.9.2 [Convex optimization and Slater's condition]**.**
Consider an optimization problem $(\mathcal{X}, f, \boldsymbol{g})$. Suppose that:

1. f is concave

2. $\boldsymbol{g}$ is concave

3. (Slater's condition) There exists an feasible relative interior point $\widehat{\boldsymbol{x}} \in \operatorname{relint}(\operatorname{feas}(\mathcal{X}, f, \boldsymbol{g}))$, i.e., $\widehat{\boldsymbol{x}} \in \operatorname{int}(\mathcal{X})$ and for all $c \in [l]$, if g_c is affine then $g_c(\widehat{\boldsymbol{x}}) \geq 0$, and $g_c(\widehat{\boldsymbol{x}}) > 0$ otherwise.

As the proof of the strong duality theorem is more involved, we state the theorem without proof and refer the reader to page 234 of Boyd et al. (2004). Briefly, the theorem is usually proven via the Separating Hyperplane Theorem.

Theorem 2.9.2 [Strong duality via Slater's condition]**.**
Consider an optimization problem $(\mathcal{X}, f, \boldsymbol{g})$ and suppose that Assumptions 2.9.1 and 2.9.2 are satisfied, then strong duality holds.

When strong duality holds, then for any solution $\boldsymbol{x}^* \in \mathcal{X}$ of the primal problem, we are guaranteed the existence of some associated optimal slack variables $\boldsymbol{\lambda}^* \in \mathbb{R}^l_+$ s.t. $\boldsymbol{\lambda}^* \in \arg\min_{\boldsymbol{\lambda} \in \mathbb{R}^l_+} \ell(\boldsymbol{x}^*, \boldsymbol{\lambda})$ which are solutions to the dual problem. More importantly, under strong duality, we can derive necessary conditions that characterize the any tuple $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) \in \mathcal{X} \times \mathbb{R}^l_+$ of primal and dual problem solutions known as the Kahn-Karush-Tucker (KKT) conditions (Kuhn and Tucker, 1951). For convenience, for the following theorem suppose that $\mathcal{X} \doteq \mathbb{R}^n$.[9]

Theorem 2.9.3 [Karush–Kuhn–Tucker theorem]**.**
Consider a Euclidean normed vector space $(\mathbb{R}^n, \|\cdot\|)$, and an associated optimization problem $(\mathcal{X}, f, \boldsymbol{g})$. Suppose that Assumption 2.9.1 and Assumption 2.9.2 hold, and that in addition f and $\boldsymbol{g}$ are locally Lipschitz continuous. Then, any tuple $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*) \in \mathcal{X} \times \mathbb{R}^l$ of primal and dual problem solutions must satisfy the following conditions:

1. (Stationarity) $\boldsymbol{0} \in \mathcal{D} f(\boldsymbol{x}^*) + \sum_{i=1}^l \lambda_i^* \mathcal{D} g_i(\boldsymbol{x}^*)$

2. (Complementary Slackness) for all $i \in [l]$, $\lambda_i^* g_i(\boldsymbol{x}^*) = 0$

3. (Primal Feasibility) $\boldsymbol{g}(\boldsymbol{x}^*) \geq \boldsymbol{0}$

[9]When $\mathcal{X}$ can be represented by finitely many (in)equality constraints, this assumption is without loss of generality, as $\mathcal{X}$ can be represented by the (in)equality constraint functions $\boldsymbol{g}$. Alternatively, if a relative interior solution to the primal problem exists, then the assumption is also without loss of generality. In addition, the KKT Theorem can be generalized to arbitrary $\mathcal{X}$, however as this more general characterization will not be used in this thesis, we omit it for simplicity.

4. (Dual Feasibility) $\boldsymbol{\lambda} \geq \mathbf{0}$

In the above theorem, if we assume that Assumption 2.9.2, holds, then the conditions are in addition sufficient.

Often, we also are interested in understanding the convexity properties of set of solutions to the primal and dual problems. The following result whose proof can be obtain combining the results in Chapter 6, Section 3 of Berge (1997) and in Proposition 4.1 of Kyparisis (1985).

Theorem 2.9.4 [Properties of primal solution set]**.**
Consider an optimization problem $(\mathcal{X}, f, \boldsymbol{g})$. Suppose that:

1. f is continuous, quasiconcave
2. $\mathrm{feas}(\mathcal{X}, f, \boldsymbol{g})$ is non-empty, compact, and convex.

Then, the set of solutions to the primal problem is non-empty, compact, and convex.

If we instead assume Assumption 2.9.2, we then also obtain a characterization of the set of solution for the primal and dual problem both (see Theorem 5 of Rockafellar (1971)).

Theorem 2.9.5 [Properties of saddle point solution set]**.**
Consider an optimization problem $(\mathcal{X}, f, \boldsymbol{g})$. Suppose that Assumption 2.9.1 and Assumption 2.9.2 hold. Then, the set of solutions to the primal and dual problem is non-empty, compact, and convex.

2.9.3 Parametric Constrained Optimization

In many problems of interest, we face parametric optimization problems, i.e., optimization problems $(\mathcal{X}, f, \boldsymbol{g})$ in which the objective $\boldsymbol{x} \mapsto f(\boldsymbol{x}; \boldsymbol{\omega})$ and $\boldsymbol{x} \mapsto \boldsymbol{g}(\boldsymbol{x}; \boldsymbol{\omega})$ constraint function depend on the value of some parameter $\boldsymbol{\omega} \in \Theta$.

Consider metric spaces $(\mathbb{R}^n, d_{\mathcal{X}})$ and $(\mathbb{R}^d, d_{\Theta})$. A **parametric (constrained) optimization problem** $(n, d, l, \Theta, \mathcal{X}, f, \boldsymbol{g})$, denoted $(\Theta, \mathcal{X}, f, \boldsymbol{g})$ when clear from context, consists of a **basic feasible set** $\mathcal{X} \subseteq \mathbb{R}^n$**, set of parameters** $\Theta \subseteq \mathbb{R}^d$, a **parametric objective function** $f : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}$, and an **inequality constraint**

function $\boldsymbol{g} : \mathbb{R}^n \times \mathbb{R}^d \to \mathbb{R}^l$, which for all $\boldsymbol{\omega} \in \Theta$ define the following maximization problem:

$$\max_{\boldsymbol{x} \in \mathcal{X}} \quad f(\boldsymbol{x}; \boldsymbol{\omega}) \tag{2.24}$$

$$\text{Constrained by} \quad \boldsymbol{g}(\boldsymbol{x}; \boldsymbol{\omega}) \geq \boldsymbol{0} \tag{2.25}$$

When faced with a parametric optimization problem, we are often interested in understanding properties of the **marginal function** $f^*(\boldsymbol{\omega}) \doteq \max_{\boldsymbol{x} \in \mathcal{X}: \boldsymbol{g}(\boldsymbol{x}) \geq \boldsymbol{0}} f(\boldsymbol{x}; \boldsymbol{g})$ and the **solution correspondence** $\mathcal{X}^*(\boldsymbol{\omega}) \doteq \arg\max_{\boldsymbol{x} \in \mathcal{X}: \boldsymbol{g}(\boldsymbol{x}) \geq \boldsymbol{0}} f(\boldsymbol{x}; \boldsymbol{g})$. Theorems that characterize properties of the marginal function and solution correspondence are known under the name of the maximum theorem. For convenience define the **constraint correspondence** $\mathcal{C}(\boldsymbol{\omega}) \doteq \{\boldsymbol{x} \in \mathcal{X} \mid \boldsymbol{g}(\boldsymbol{x}; \boldsymbol{\omega}) \geq \boldsymbol{0}\}$.

Theorem 2.9.6 [Maximum Theorem for Continuity]**.**
Consider a **parametric optimization problem** $(\Theta, \mathcal{X}, f, \boldsymbol{g})$. Suppose that 1) the constraint correspondence $\mathcal{C}$ is continuous and non-empty, compact-valued, and 2) f is continuous, then the following hold:

1. f^* is continuous, and
2. $\mathcal{X}^*$ is upper hemicontinuous, and non-empty, and compact-valued.

We note that the continuity cf the constraint correspondence $\mathcal{C}$ can be guaranteed under either of the following assumptions by Theorem 5.9 and Example 5.10 of Rockafellar and Wets (2009):

Assumption 2.9.3 [Continuity via quasiconvex representation]**.**
Consider a correspondence $\mathcal{C}(\boldsymbol{\omega}) \doteq \{\boldsymbol{x} \in \mathcal{X} \mid \boldsymbol{g}(\boldsymbol{x}; \boldsymbol{\omega}) \geq \boldsymbol{0}\}$, and suppose that:

1. $\mathcal{X}$ is compact
2. $\boldsymbol{g}$ is continuous and quasiconcave

Assumption 2.9.4 [Continuity via Slater's condition]**.**
Consider a correspondence $\mathcal{C}(\boldsymbol{\omega}) \doteq \{\boldsymbol{x} \in \mathcal{X} \mid \boldsymbol{g}(\boldsymbol{x}; \boldsymbol{\omega}) \geq \boldsymbol{0}\}$, and suppose that

1. $\mathcal{X}$ is compact
2. $\boldsymbol{g}$ is continuous and concave
3. (Slater's condition) For all $\boldsymbol{\omega} \in \Theta$, there exists a feasible relative interior point $\widehat{\boldsymbol{x}} \in \operatorname{relint}(\mathcal{C}(\boldsymbol{\omega}))$, i.e., $\widehat{\boldsymbol{x}} \in \operatorname{int}(\mathcal{X})$ and for all $c \in [l]$, if $\boldsymbol{x} \mapsto g_c(\boldsymbol{x}, \boldsymbol{\omega})$ is affine then $g_c(\widehat{\boldsymbol{x}}, \boldsymbol{\omega}) \geq 0$, and $g_c(\widehat{\boldsymbol{x}}, \boldsymbol{\omega}) > 0$ otherwise.

Going further than continuity, we might more generally be interested in understanding convexity and concavity properties of the marginal function and solution correspondence. As a corollary, of Theorem 2.9.4 we have the following convexity characterization of the image of the solution correspondence.

Corollary 2.9.1 [Convex-valued solution correspondence]**.**
Consider a **parametric optimization problem** $(\Theta, \mathcal{X}, f, \boldsymbol{g})$. Suppose that 1) the constraint correspondence $\mathcal{C}$ is non-empty-, compact-, and convex-valued, and 2) for all $\boldsymbol{\omega} \in \Theta$, $\boldsymbol{x} \mapsto f(\boldsymbol{x}; \boldsymbol{\omega})$ is continuous and quasiconcave, then $\mathcal{X}^*$ is non-empty-, and compact-, and convex-valued.

The following theorem provides sufficient conditions for the concavity of the marginal function and convexity of the solution correspondence (see Theorem 2.1 and 3.1 of Kyparisis and Fiacco (1987)).

Theorem 2.9.7 [Convexity/concavity of marginal function]**.**
Consider a **parametric optimization problem** $(\Theta, \mathcal{X}, f, \boldsymbol{g})$. Suppose that 1) the constraint correspondence $\mathcal{C}$ is convex, and 2) f is concave, then f^* and $\mathcal{X}^*$ are concave.

Part I

Variational Inequalities and Walrasian Economies

Chapter 3

Scope and Motivation

3.1 Scope

Part I of this thesis, is divided into three chapters. In Chapter 4, after reviewing background material on variational inequalities we[1] will introduce two new type of methods with polynomial-time convergence guarantees. The first type of methods will be a family of first-order methods known under the name of the mirror extragradient method. We will prove that this method converges to a strong solution of any variational inequality for which a weak solution exists. Further, in the absence of a weak solution, we will prove local convergence to a strong solution when the algorithm's first iterate is initialized close enough to a local weak solution. As first order methods are not guaranteed to converge beyond setting where a (local) weak solution exists, we will then turn my attention to a class of second-order methods known as merit function methods. In particular, we will introduce the primal mirror descent which we will show is guaranteed to converge to a local minimum of the regularized primal gap function of any Lipschitz-smooth variational inequality.

In Chapter 5, after reviewing background material on Walrasian economies, we will show that the set of Walrasian equilibria of any Walrasian economy is equal to set of strong solutions of an associated variational inequality. In addition, we will show that the gradient method applied to this variational inequality is equivalent to solving the Walrasian economy via a well-known price-adjustement process known as *tâtonnement*. Further, running the mirror extragradient method on this variational inequality, we obtain a

[1]The works in Part I of this thesis were a collaboration with Amy Greenwald, with Sadie Zhao additionally contributing to the verification of Chapter 6.

new family of price adjustment processes called the mirror *extratâtonnement* process for which we show polynomial-time convergene to a Walrasian equilibrium in a large class of Walrasian economies. Finally, using the VI characterization of Walrasian equilibrium, we will introduce a class of merit function methods with polynomial-time convergence guarantees for an even broader class Walrasian economies.

While the results provided in Chapter 5 answer a number of open questions, due to the high generality of the variational inequality framework, they often might not depict accurately the convergence behavior of price-adjustment processes in specific types of Walrasian economies used in practice. To illustrate this point, in Chapter 6, using the tools of convex optimization and consumer theory, we will provide a more fine-grained convergence analysis with better guarantees for a particular *tâtonnement* process in a class of Walrasian economies used widely in practice, known as Fisher markets. In particular, we will show the sublinear convergence of entropic *tâtonnement* to a Walrasian equilibrium in homothetic Fisher markets with bounded elasticity of Hicksian demand.

3.2 Motivation

Walrasian economies (or general equilibrium models), first studied by French economist Léon Walras in 1874, are a broad mathematical framework for modeling any economic system governed by supply and demand (Walras, 1896). A Walrasian economy consists of a finite set of commodities, characterized by an excess demand function that maps values for commodities, called **prices**, to positive (resp. negative) quantities of each commodity demanded (resp. supplied) in excess. Walras proposed a steady-state solution of his economy, namely a Walrasian (or competitive) equilibrium, represented by a collection of per-commodity prices which is **feasible**, i.e., there is no excess demand for any commodity, and for which **Walras' law** holds, i.e., the value of the excess demand is equal to 0.

Walras did not establish conditions ensuring the existence of an equilibrium, leaving the question unresolved until the 1950s (Arrow and Debreu, 1954), but argued, albeit without conclusive evidence, that his economy would settle at a Walrasian equilibrium via a **price-adjustment process** (i.e., any process that generates a sequence of prices based on prior prices and associated excess demands), known as ***tâtonnement***, which mimics the behavior of the **law of supply and demand**, updating prices at a rate equal to the excess demand (Walras, 1896; Uzawa, 1960; Arrow and Hurwicz, 1958). To motivate the relevance of *tâtonnement* to real-world economies, Walras argued that *tâtonnement* is a **natural price-adjustment process**, in the sense that if each each commodity is owned by a different seller, then each seller can update the price of its commodity without coordinating with other sellers, using only information about the excess demand of its commodity, hence making it plausible that *tâtonnement* could explain the movement of prices in real-world economies where sellers again do not coordinate with one another.

Nearly half a century after Walras' initial foray into general equilibrium analysis, a group of academics brought together by the Cowles Commission in 1939 reinitiated a study of Walras' economic model with the purpose of bringing rigorous mathematics to the analysis of markets. One of the earliest and most important outputs of this collaborative effort was the introduction of a broad and well-justified class of Walrasian economies known as **competitive economies** (Arrow and Debreu, 1954), for which the existence of Walrasian equilibrium was established by a novel application of fixed point theorems to economics. With the question of existence thus resolved, the field subsequently turned its focus to investigating questions on the **stability** of Walrasian equilibrium i.e., which price-adjustment processes can settle at a Walrasian equilibrium and under what assumptions? (Uzawa, 1960; Balasko, 1975; Arrow and Hurwicz, 1958; Cole and Fleischer, 2008; Cheung et al., 2018; 2013; Jain et al., 2005; Codenotti et al., 2005; 2006; Chen and Teng, 2009).

Most relevant work on stability has been concerned with the convergence properties of *tâtonnement*. Beyond Walras' justification for *tâtonnement*'s relevance to real-world economies, research on *tâtonnement* in the post-world war II economics literature is motivated by the fact that it can be understood as a plausible explanation of how prices move in real-world markets (Gillen et al., 2020). Hence, if one could prove that *tâtonnement* is a **universal price-adjustment process** (i.e., a price-adjustment process that converges to a Walrasian equilibrium in all competitive economies), then perhaps it would be justifiable to claim real-world economies would also eventually settle at a Walrasian equilibrium.

In 1958, Arrow and Hurwicz (1958) established the convergence of a continuous-time variant of *tâtonnement* in Walrasian economies with an excess demand function satisfying the weak axiom of revealed preferences (WARP) (Afriat, 1967), which among others, includes Walrasian economies satisfying the GS condition

(Arrow et al., 1959; Arrow and Hurwicz, 1960). This result was complemented by Nikaidô and Uzawa's (Nikaidô and Uzawa, 1960) result on the convergence of a discrete-time variant of *tâtonnement* in Walrasian economies satisfying WARP—albeit without any non-asymptotic convergence guarantees. These initial results sparked hopes that *tâtonnement* could be a universal price-adjustment process.

Furthermore, as there in general exists no closed-form formulas for Walrasian equilibria, these results ignited further interest in discovering algorithms to compute a Walrasian equilibrium, as *tâtonnement* could be implemented on a computer to obtain numerical approximations of Walrasian equilibria in Walrasian economies. Indeed, these early results on the stability of *tâtonnement* inspired a new line of work on **applied general equilibrium** (Scarf, 1967b;a; Scarf and Hansen, 1973; Scarf, 1982) initiated by Herbert Scarf (Arrow and Kehoe, 1994), whose goal was to establish "a general method for the explicit numerical solution of the neoclassical [Walrasian economy] model" (Scarf and Hansen, 1973). The motivation behind this research agenda was a desire to predict the impact of economic policy on an economy by estimating the parameters of a parametric Walrasian economy from empirical data, and then running a comparative static analysis to compare the numerical solution of the Walrasian economy before and after the implementation of the policy.

Unfortunately, soon after initiating this research agenda, Scarf dashed all hopes that *tâtonnement* could be a universal price-adjustment process by showing that the sequence of prices generated by a continuous-time variant of *tâtonnement* can cycle ad infinitum around the Walrasian equilibrium of his eponymous competitive economy, with only three commodities and an excess demand function generated by three consumers with Leontief preferences, i.e., **the Scarf economy** (Scarf, 1960). Even more discouragingly, when applied to the Scarf economy, the prices generated by discrete-time variants of *tâtonnement* spiral away from the Walrasian equilibrium, moving further and further away from equilibrium.

Scarf's negative result seems to have discouraged further research by economists on the stability of Walrasian equilibrium (Fisher, 1975). Despite research on this question coming to a near halt, one positive outcome was achieved, on the convergence of a non-*tâtonnement* update rule known as **Smale's process** (Herings, 1997; Kamiya, 1990; van der Laan et al., 1987; Smale, 1976), which updates prices at the rate of the product of the excess demand and the inverse of its Jacobian, to a Walrasian equilibrium in competitive economies which have an excess demand that has a non-singular Jacobian, including Scarf economies. Unfortunately, this convergence result for Smale's process comes with two caveats: 1) Smale's process is not a "natural" price-adjustment process, as it updates the price of each commodity using information about not only the excess demand of the commodity but also the derivative of the excess demand function with respect to each commodity in the economy, 2) convergence of discrete time-variants of Smale's process require the excess demand to satisfy the law of supply and demand, which even Walrasian economies that satisfy the GS or WARP conditions do not satisfy.

Nearly half a century after these seminal analyses of competitive economies, research on the stability and efficient computation of Walrasian equilibrium is once again coming to the fore, motivated by applications of algorithms to compute Walrasian equilibrium in dynamic stochastic general equilibrium models in macroeconomics (Geanakoplos, 1990; Sargent and Ljungqvist, 2000; Taylor and Woodford, 1999; Fernández-Villaverde, 2023), and the use of algorithms such as *tâtonnement* to solve models of transactions on crypotocurrency blockchains (Leonardos et al., 2021; Liu et al., 2022; Reijsbergen et al., 2021) and load balancing over networks (Jain et al., 2013). In contrast to the prior literature on the stability of *tâtonnement*, which was primarily concerned with proving asymptotic convergence of price-adjustment processes to a Walrasian equilibrium, this line of work is also concerned with obtaining non-asymptotic convergence rates, and hence computing approximate Walrasian equilibria in polynomial time.

The first result on this question is due to Codenotti et al. (2005), who introduced a discrete-time version of *tâtonnement*, and showed that in exchange economies that satisfy **weak gross substitutes (WGS)** (i.e., the excess demand of any commodity *weakly* increases if the price of any other commodity increases, fixing all other prices), the *tâtonnement* process converges to an approximate Walrasian equilibrium in a number of steps which is polynomial in the inverse of the approximation factor and size of the problem. Unfortunately, soon after this positive result appeared, Papadimitriou and Yannakakis (2010) showed that it is impossible for a price-adjustment process based on the excess demand function to converge in polynomial time to a Walrasian equilibrium in general, ruling out the possibility of Smale's process (and many others), justifying the notion of Walrasian equilibrium in all competitive economies. Nevertheless, further study of the convergence of price-adjustment processes such as *tâtonnement* under stronger assumptions, or in simpler models than full-blown Arrow-Debreu competitive economies, continues, as these processes are being deployed in practice (Jain et al., 2013; Leonardos et al., 2021; Liu et al., 2022; Reijsbergen et al., 2021).[2]

3.3 Contributions

3.3.1 A Tractable Variational Inequality Framework for Walrasian Economies

To address the challenge brought forward by the impossibility result of Papadimitriou and Yannakakis (2010), we provide a characterization of Walrasian equilibrium using the variational inequality (VI) optimization framework. To this end, we first introduce the class of **mirror extragradient algorithms** and prove the polynomial-time convergence of this method for VIs thta satisfy a computational tractability condition known as the Minty condition (Minty, 1967), and a generalization of Lipschitz-continuity known as Bregman-continuity. A Bregman-continuous (or relatively continuous (Lu, 2019)) function is one for which the change in the Euclidean distance of the function between any two points is proportional to Bregman divergence between those two points.

With these tools in place, we then demonstrate that the set of Walrasian equilibria of **balanced economies**—those Walrasian economies with an excess demand that is homogeneous of degree 0, and satisfy weak Walras's (i.e., the value of the excess demand is less than or equal to 0 at any price)—a class of Walrasian economies which among others includes Arrow and Debreu's competitive economies (Arrow and Debreu, 1954), is equal to the set of strong solutions of a VI that satisfies the Minty condition (Minty, 1967). With this characterization in hand, we apply the mirror extragradient algorithm to solving this VI, which gives rise to a novel natural price-adjustment process we call **mirror *extratâtonnement***.

An important property of the VI we introduce is that its search space for prices is *not* restricted to the unit simplex as it is traditionally the case for competitive economies, but rather to the unit box. This fact offers us insight into understanding how we can overcome Papadimitriou and Yannakakis's impossibility result on the exponential-time convergence of price-adjustment processes in general Walrasian economies. Papadimitriou and Yannakakis's definition of a price-adjustment process restricts prices generated by the process to lie within the unit simplex; however, when the search space of the VI we introduce is restricted in

[2]We refer the reader to Sections 4.2.2 to 5.3.2 for additional related works on algorithms for Walrasian Economies and VIs.

this way, the VI fails to satisfy the Minty condition, and is thus computationally intractable. This suggests that relaxing the requirement that prices lie within the unit simplex can overcome the challenge of the exponential-time convergence of price-adjustment processes in Walrasian economies, and allow for the efficient computation of Walrasian equilibrium, at least in practice.

The reader might wonder what we mean by "in practice". As it turns out the VI characterization we provide is in general discontinuous at one point in its search space, namely when the prices for all commodities are 0. As such, because it is not possible to ensure the Lipschitz-continuity or Bregman-continuity of the excess demand on the unit box in general, it is not possible to obtain polynomial-time convergence of our mirror extragradient to solve our VI without further assumptions. Nevertheless, as we discuss in the sequel, we observe the fast convergence of mirror *extratâtonnement* process in a large class of competitive economies, including very large instances with Leontief consumers, for which the computation of a Walrasian equilibrium is known to be PPAD-complete (Codenotti et al., 2006; Deng and Du, 2008). This suggests the need for a novel assumption that would explain the convergence of process to a Walrasian equilibrium in practice. To this end, we introduce the pathwise Bregman-continuity assumption, a condition that requires the excess demand to be Bregman-continuous along the sequence of prices generated by the mirror *extratâtonnement* process, which we show is sufficient to guarantee the polynomial-time convergence of our process.

While the pathwise Bregman-continuity assumption provides intuition on the fast convergence of the mirror *extratâtonnement* processes in practice, it is hard to very this assumption in advance. Thus, we subsequently restrict our search space for prices to the unit simplex, and restrict our attention to competitive economies that are variationally stable on the unit simplex (i.e., those economies for which the associated VI satisfies the Minty condition) and have a **bounded elasticity of excess demand** (i.e., the percentage change in the excess demand for a percentage change in prices is bounded across all price changes). We demonstrate that under these additional assumptions, the VI is guaranteed to satisfy the Minty condition, and show that for such economies the excess demand is Bregman-continuous, thus providing the first polynomial-time convergence result for a price adjustment processes in this class of Walrasian economies, which among others includes competitive economies that satisfy WGS, and more generally, WARP.

3.3.2 Variational Inequalities

Our first major contribution is introducing the class of mirror extragradient algorithms, a generalization of Korpelevich's extragradient method (Korpelevich, 1976) for solving VIs. We establish best-iterate convergence of the class of mirror extragradient algorithms to a ε-strong solution of VIs that satisfy the Minty condition and are Bregman-continuous in $O(1/\varepsilon^2)$ evaluations of the optimality operator of the VI (Theorem 4.3.1). Our result generalizes the results and proof techniques of Huang and Zhang (2023) for the extragradient method, and extends the convergence results of Zhang and Dai (2023) for the unconstrained mirror extragradient method to constrained domains. In addition, to provide further justification for the convergence of the mirror extratâtonnement process in balanced economies, we establish suitable conditions for the local convergence of the mirror extragradient algorithm to an ε-strong solution of any Bregman-continuous VI that does *not* satisfy the Minty condition—to the best of our knowledge, the first result of its kind (Theorem 4.3.2).

3.3.3 Walrasian Economies

While a characterization of the set of Walrasian equilibria of any Walrasian economy as the solution set of an associated complementarity problem (i.e., a VI where the constraint set is the positive orthant) seems to have already been known (Dafermos, 1990), for balanced economies, we provide the first computationally tractable characterization of Walrasian equilibria as the set of strong solutions of a VI that satisfies the Minty condition and whose constraint set is given by the unit box. We then apply the mirror extragradient method to obtain a novel natural price-adjustment process we call the mirror *extratâtonnement* process (Algorithm 6), and prove its convergence in all balanced economies that satisfy pathwise Bregman-continuity (Corollary 5.4.1).

We then restrict our attention to a novel class of competitive economies, namely those which are variationally stable on the unit simplex, and establish the polynomial-time convergence of the mirror *extratâtonnement* process in all such economies assuming bounded elasticity of excess demand (Theorem 5.4.2). Our convergence result also provides the first polynomial-time convergence result for price-adjustment processes in the class of economies that satisfy WARP, and generalizes the well-known *tâtonnement* convergence result in competitive economies with bounded elasticity of excess demand that satisfy WGS (Codenotti et al., 2005).

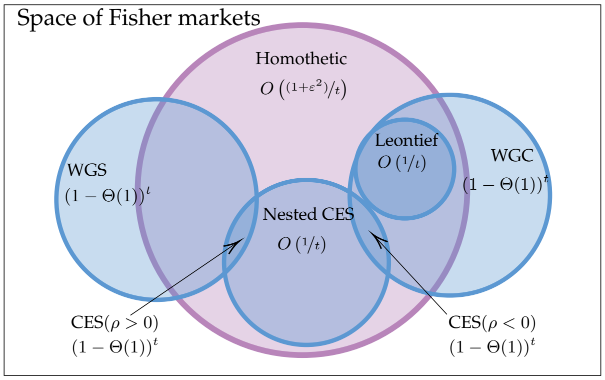
We then apply the mirror *extratâtonnement* process to the Scarf economy, and prove its polynomial-time convergence to the unique Walrasian equilibrium of the economy (Corollary 5.4.4). As such, the mirror *extratâtonnement* process is the first discrete-time *natural* price adjustment process to converge in the Scarf economy.

Finally, we run a series of experiments on a variety of competitive economies where we verify that the pathwise Bregman-continuity assumption holds, and demonstrate that our algorithm converges to a Walrasian equilibrium at the rate predicted by our theory. Importantly, our experiments include examples of randomly initialized very large competitive economies (~ 500 consumers and ~ 500 commodities) which are known to be PPAD-complete (e.g., Leontief economies), for which we show that our algorithm computes a Walrasian equilibrium fast without failure in all cases.

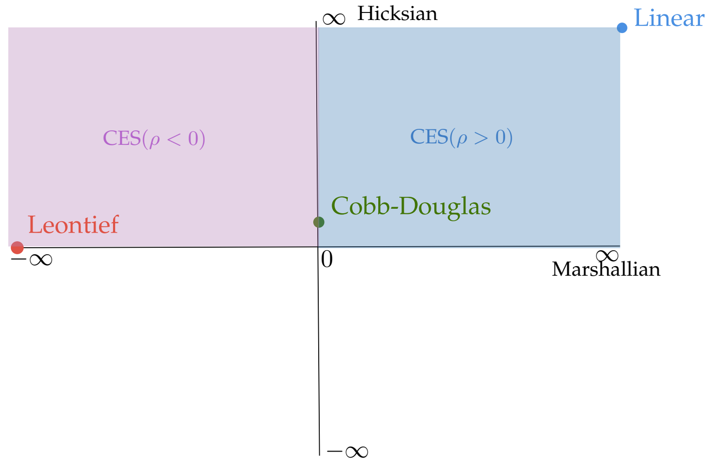
3.3.4 Fisher Markets

Earlier work (Cheung, 2014; Cheung et al., 2013) has established a convergence rate of $(1 - \Theta(1))^T$ for CES Fisher markets excluding the linear and Leontief cases, and of $O\left(1/T\right)$ for Leontief and nested[3] CES Fisher markets, where $T \in \mathbb{N}_+$ is the number of iterations for which *tâtonnement* is run. In linear Fisher markets, however, *tâtonnement* does not converge. We generalize these results by proving a convergence rate of $O((1+\epsilon^2)/T)$, where ϵ is the maximum absolute value of the price elasticity of Hicksian demand across all buyers. Our convergence rate covers the full spectrum of homothetic Fisher markets, including mixed CES markets, i.e., CES markets with linear, Leontief, and (nested) CES buyers, unifying previously existing disparate convergence and non-convergence results. In particular, for $\epsilon = 0$, i.e., Leontief Fisher markets, we recover the best-known convergence rate of $O(1/T)$, and as $\epsilon \to \infty$, i.e., linear Fisher markets, we obtain the non-convergent behaviour of *tâtonnement* (Cole and Tao, 2019). We summarize known convergence results in light of our results in Figure 3.1a.

[3] See Chapter 10 of Cheung (2014).



(a) The convergence rates of *tâtonnement* for different Fisher markets. We color previous contributions in blue, and our contribution in red, i.e., we study homothetic Fisher markets where ϵ is the maximum absolute value of the price elasticity of Hicksian demand across all buyers. We note that the convergence rate for WGS markets does not apply to markets where the price elasticity of Marshallian demand is unbounded, e.g., linear Fisher markets; likewise, the convergence rate for nested CES Fisher markets does not apply to linear or Leontief Fisher markets.



(b) Cross-price elasticity taxonomy of well-known homogeneous utility functions. There are no previously studied utility functions in the space of utility functions with negative Hicksian cross-price elasticity. Future work could investigate this space and prove faster convergence rates than those provided in this paper. We note that our convergence result covers the entire spectrum of this taxonomy (excluding limits of the y-axis).

Figure 3.1: A summary of known results in Fisher markets.

We observe that, in contrast to general competitive economies, in homothetic Fisher markets, concavity of the utility functions is not necessary for the existence of competitive equilibrium (Theorem 6.2.1). A computational analog of this result also holds, namely that *tâtonnement* converges in homothetic Fisher markets, even when buyers' utility functions are non-concave. Our results parallel known results on the convergence of *tâtonnement* in WGS markets, where concavity of utility functions is again not necessary for convergence (Codenotti et al., 2005).

Chapter 4

Variational Inequalities

4.1 Background

Variational inequalities (Facchinei and Pang, 2003) are a mathematical modeling framework whose study dates back to the early 1960s (Lions and Stampacchia, 1967; Hartman and Stampacchia, 1966; Browder, 1965; Grioli, 1973; Brezis and Sibony, 2011). Their utility lies in their very broad mathematical formulation which allows one to solve other mathematical modeling problems using the tools of functional analysis. They have found a great number of applications to problems in engineering and finance (Facchinei and Pang, 2003) over the years, and have seen an increased interest due to their novel applications in machine learning, to problems ranging from the training of generative adversarial neural networks (Goodfellow et al., 2014) to robust optimization (Ben-Tal et al., 2009).

4.1.1 Stampacchia Variational Inequality

Consider an inner product space $(\mathcal{U}, \langle \cdot, \cdot \rangle)$. A **(generalized[1]) variational inequality (VI)**, denoted $(\mathcal{X}, \mathcal{F})$, consists of a **constraint set** $\mathcal{X} \subseteq \mathcal{U}$ and an **optimality operator** $\mathcal{F} : \mathcal{U} \rightrightarrows \mathcal{U}^*$. For notational convenience, for any $\boldsymbol{x} \in \mathcal{X}$, we denote any arbitrary element of $\mathcal{F}(\boldsymbol{x})$ by $\boldsymbol{f}(\boldsymbol{x})$, and denote the variational inequality by $(\mathcal{X}, \boldsymbol{f})$ if $\mathcal{F}$ is singleton-valued.

[1]When $\mathcal{F}$ is singleton-valued a generalized variational inequality is simply called a variational inequality. As our computational results will be limited to generalized variational inequalities where $\mathcal{F}$ is singleton-valued, for simplicity, we will refer to generalized variational inequalities simply as variational inequalities.

Any VI $(\mathcal{X}, \mathcal{F})$ defines a problem known as the **(generalized) Stampacchia variational inequality (SVI)** (Lions and Stampacchia, 1967):

$$\text{Find } \boldsymbol{x}^* \in \mathcal{X} \text{ such that} \qquad \langle \boldsymbol{f}(\boldsymbol{x}^*), \boldsymbol{x} - \boldsymbol{x}^* \rangle \geq 0 \qquad \text{for all } \boldsymbol{x} \in \mathcal{X} \tag{4.1}$$

$$\text{and for some} \qquad \boldsymbol{f}(\boldsymbol{x}^*) \in \mathcal{F}(\boldsymbol{x}^*) \tag{4.2}$$

A solution to a SVI is called a **strong solution** of the variational inequality $(\mathcal{X}, \mathcal{F})$. Just like in convex optimization settings (see Section 1.1.2 of Nesterov (1998)), in practice, it is not possible to compute an exact strong solution to a VI $(\mathcal{X}, \mathcal{F})$, and as such we have to resort to approximate solutions which we call the ε-strong solution. Note that in the following definition, in line with the literature (see, for instance Section 1.2 of Diakonikolas (2020)), the inequality is negated (and as such inverted).

Definition 4.1.1 [Strong Solution]**.**
Given an **approximation parameter** $\varepsilon \geq 0$, a ε-**strong** (or **Stampacchia**) **solution** of the VI $(\mathcal{X}, \mathcal{F})$ is a $\boldsymbol{x}^* \in \mathcal{X}$ that satisfies the following:

$$\exists \boldsymbol{f}(\boldsymbol{x}^*) \in \mathcal{F}(\boldsymbol{x}^*), \qquad \max_{\boldsymbol{x} \in \mathcal{X}} \langle \boldsymbol{f}(\boldsymbol{x}^*), \boldsymbol{x}^* - \boldsymbol{x} \rangle \leq \varepsilon \tag{4.3}$$

A 0-strong solution is simply called a **strong solution**. We denote the set of ε-strong (resp. the set of strong) solutions a VI $(\mathcal{X}, \mathcal{F})$ by $\mathcal{SVI}_\varepsilon(\mathcal{X}, \mathcal{F})$ (resp. $\mathcal{SVI}(\mathcal{X}, \mathcal{F})$).

A large number of mathematical optimization problems can be cast as VI problems, and as such they have found a large number of applications. We will explore a number of these applications to Game Theory in this thesis, and mention now only a simple application to convex optimization for illustrative purposes.

Example 4.1.1 [Convex Optimization as a VI]**.**
Consider a convex optimization problem:

$$\min_{\boldsymbol{x} \in \mathcal{X}} h(\boldsymbol{x})$$

where $\mathcal{X} \subseteq \mathcal{U}$ is a non-empty, compact, and convex constraint set, and $h : \mathcal{U} \to \mathbb{R}$ is the continuous, and convex objective function.

Any ε-minimum $\boldsymbol{x}^* \in \mathcal{X}$ s.t. $h(\boldsymbol{x}^*) - \min_{\boldsymbol{x} \in \mathcal{X}} h(\boldsymbol{x}) \leq \varepsilon$ of the above problem satisfies the following necessary and sufficient optimality conditions (see, for instance section 2 of Crespi et al. (2005)):

$$\langle \partial h(\boldsymbol{x}^*), \boldsymbol{x}^* - \boldsymbol{x} \rangle \leq \varepsilon \qquad\qquad \forall \boldsymbol{x} \in \mathcal{X}, \exists \partial h(\boldsymbol{x}^*) \in \mathcal{D}h(\boldsymbol{x}^*)$$

Taking a maximum over $\boldsymbol{x} \in \mathcal{X}$, we can then see that the set of ε-minima of $(\mathcal{X}, h)$, corresponds to the set of ε-strong solutions $\mathcal{SVI}_\varepsilon(\mathcal{X}, \mathcal{D}h)$ of the VI $(\mathcal{X}, \mathcal{D}h)$ where the optimality operator is given by the subdifferential correspondence of the objective h in the convex optimization problem.

Going further, if $(\mathcal{X}, h)$ is not a convex optimization problem but instead a weakly-convex optimization problem, then the optimality conditions are only sufficient, in which case the set of ε-strong solutions $\mathcal{SVI}_\varepsilon(\mathcal{X}, \mathcal{D}h)$ of the VI $(\mathcal{X}, \mathcal{D}h)$ is called the set of ε-**stationary points** of the optimization problem.

Strong solutions can be shown to exist in a broad of class known as continuous.

Definition 4.1.2 [Continuous VIs]**.**
A **continuous** VI is a VI $(\mathcal{X}, \mathcal{F})$ such that:

1. $\mathcal{X}$ is non-empty, compact, and convex
2. $\mathcal{F}$ is upper hemcontinuous, non-empty-, compact-, and convex-valued

The proof of existence of a strong solution in continuous VIs relies on a fixed-point argument applied to a mapping whose fixed points correspond to strong solutions of the VI, whose fixed points can in turn be shown to exist by the Glicksberg-Kakutani fixed point theorem (see Theorem 2.4.1). To avoid introducing additional cumbersome notation, we refer the reader to Theorem 2.2.1 of Facchinei and Pang (2003) for a reference.

Theorem 4.1.1 [Existence of Strong Solution (Theorem 2.2.1 of Facchinei and Pang (2003))]**.**
Consider a continuous VI $(\mathcal{X}, \mathcal{F})$, then there exists at least one strong solution.

4.1.2 Minty Variational Inequality

An alternative but related problem formulation for VIs is the **(generalized) Minty variational inequality (MVI)** (Minty, 1967). Given a VI $(\mathcal{X}, \mathcal{F})$, the MVI is defined as:

$$\text{Find } \boldsymbol{x}^* \in \mathcal{X} \text{ such that} \qquad \langle f(\boldsymbol{x}), \boldsymbol{x}^* - \boldsymbol{x} \rangle \leq 0 \qquad \exists \boldsymbol{x} \in \mathcal{X}, \boldsymbol{f}(\boldsymbol{x}^*) \in \mathcal{F}(\boldsymbol{x}^*) \tag{4.4}$$

A solution to a MVI is called the weak solution, for which similarly, we can define an approximate variant for computational purposes.

Definition 4.1.3 [Weak (or Minty) Solution]**.**
Given a VI $(\mathcal{X}, \mathcal{F})$ and an **approximation parameter** $\varepsilon \geq 0$, a ε**-weak (or Minty) solution** is a $\boldsymbol{x}^* \in \mathcal{X}$ that satisfies the following:

$$\exists \boldsymbol{f}(\boldsymbol{x}) \in \mathcal{F}(\boldsymbol{x}), \qquad \max_{\boldsymbol{x} \in \mathcal{X}} \langle \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x}^* - \boldsymbol{x} \rangle \leq \varepsilon \tag{4.5}$$

A 0-weak solution to the VI is simply called a **weak solution**. We denote the set of ε-weak (resp. the set of weak) solutions a VI $(\mathcal{X}, \mathcal{F})$ by $\mathcal{MVI}_\varepsilon(\mathcal{X}, \mathcal{F})$ (resp. $\mathcal{MVI}(\mathcal{X}, \mathcal{F})$).

In continuous VIs (see Definition 4.1.2), the set of weak solutions is a subset of set of strong solutions, i.e., the MVI is a refinement of the SVI. However, we note that a weak solution is in general not guaranteed to exist in continuous VIs.

Remark 4.1.1 [Strong vs. Weak Solutions]**.**
It might seem like a misnomer that the solutions of the MVI are called weak solutions as they are a subset of the strong solutions; however, beyond finite dimensional settings (e.g., the Euclidean space setting we are in), the set of weak and strong solutions are not guaranteed to have a non-empty intersection and can be totally unrelated. As a VI is a mathematical framework for modeling other mathematical problems using the tools of functional analysis, a weak solution should be interpreted as "weak" from a modeling perspective, in the following sense.

In any VI problem $(\mathcal{X}, \mathcal{F})$, we are given an optimality operator $\mathcal{F}$ which maps any vector $\boldsymbol{x} \in \mathcal{X}$ to a set of optimality conditions $\mathcal{F}(\boldsymbol{x})$. These optimality conditions are then tested against other vectors $\boldsymbol{x}' \in \mathcal{X}$ using an inner product. For strong solutions of the VI, the candidate solution vector we are seeking to compute, say $\boldsymbol{x}^*$ is mapped by the optimality operator $\mathcal{F}$; as such, the operator has to be well-defined for $\boldsymbol{x}^*$, and more importantly the optimality conditions $\mathcal{F}(\boldsymbol{x}^*)$ of the candidate solution $\boldsymbol{x}^*$ have to be directly satisfied by the candidate solution. In contrast, for the weak solutions of the VI, the optimality conditions $\mathcal{F}(\boldsymbol{x}^*)$ for $\boldsymbol{x}^*$ are not required to be satisfied, and might not even be defined (i.e., $\mathcal{F}(\boldsymbol{x}^*)$ might be empty).

That is, a weak solution is weak as there is no notion of optimality that can be attributed to it without the other test vectors, while a strong solution is strong as its optimality can almost be determined independently of the other vectors. As such, when we reduce a mathematical problem to a VI problem, the set of weak solutions might be underspecified and too broad to gain anything from casting the original problem as a VI. To see this concretely, recall Example 4.1.1, and suppose that the objective of the optimization problem is non-convex. The set of weak solutions to this VI corresponds to the set of global solutions of the optimization problem (see, for instance, Proposition 2.2 of Crespi et al. (2005)), while the set of strong solutions correspond to stationary points. This then makes the MVI formulation of the optimization problem no more informative than the original optimization problem. In contrast, the stationary points of the optimization problem, which correspond to the strong solutions of the VI provide additional information on the necessary conditions needed to be satisfied by the solutions of the original optimization problem. Perhaps, more importantly, the optimality of a local solution to the optimization problem (and as such of the associated SVI problem) can be determined almost independently of other vectors: first-order conditions fully characterize a local solution. In contrast, the optimality of a global solution to the optimization problem (and as such of the associated MVI) requires one to test the vector against all other test vectors. For additional discussion, we refer the reader to Zang and Avriel (1975).

(Theorem 4.1.1). Additionally, if we assume that the optimality operator $\mathcal{F}$ is monotone, then the set of strong and weak solutions are equal. Surprisingly, a ε-weak-solution is not guaranteed to be a ε-strong solution. However, if $\mathcal{F}$ is assumed to be monotone then any ε-strong solution is also a ε-weak solution but not vice versa. With this observation made, some remarks are in order.

4.1.3 Generalized Monotonicity Properties of Variational Inequalities

The following additional properties of VIs will be relevant in the sequel, and will define important properties of the set of strong and weak solutions of VIs. In particular, a large number of VIs satisfy a number of monotonicity conditions which makes them analytically more tractable.

Definition 4.1.4 [Monotone, Pseudomonotone and Quasimonotone VIs]**.**
A VI $(\mathcal{X}, \mathcal{F})$ is **monotone** (resp. **pseudomonotone** / **quasimonotone**) iff the optimality operator $\mathcal{F}$ is monotone (resp. **pseudomonotone** / **quasimonotone**).[2]

Another common and more property for the analysis of VIs, known as the Minty condition, is simply the existence of a Minty solution.

Definition 4.1.5 [Minty's Condition]**.**
A VI $(\mathcal{X}, \mathcal{F})$ satisfies the **Minty condition** iff $\mathcal{MVI}(\mathcal{X}, \mathcal{F}) \neq \emptyset$

With these definitions in order, we summarize the following known properties of the solution sets of VIs.

Remark 4.1.2 [Solution Set Properties]**.**
Let $\varepsilon \geq 0$, then the following implications hold:

- $(\mathcal{X}, \mathcal{F})$ is continuous $\implies \mathcal{SVI}(\mathcal{X}, \mathcal{F}) \neq \emptyset$ (Theorem 2.2.1 of Facchinei and Pang (2003)))
- $(\mathcal{X}, \mathcal{F})$ is continuous $\implies \mathcal{MVI}(\mathcal{X}, \mathcal{F}) \subseteq \mathcal{SVI}(\mathcal{X}, \mathcal{F})$
- $(\mathcal{X}, \mathcal{F})$ is monotone $\implies \mathcal{SVI}_\varepsilon(\mathcal{X}, \mathcal{F}) \subseteq \mathcal{MVI}_\varepsilon(\mathcal{X}, \mathcal{F})$
- $(\mathcal{X}, \mathcal{F})$ is pseudomonotone $\implies \mathcal{SVI}(\mathcal{X}, \mathcal{F}) \subseteq \mathcal{MVI}(\mathcal{X}, \mathcal{F})$
- $(\mathcal{X}, \mathcal{F})$ is quasimonotone with $\mathcal{X}$ non-empty, and compact $\implies \mathcal{MVI}(\mathcal{X}, \mathcal{F}) \neq \emptyset$ (Lemma 3.1 of (He, 2017))
- Suppose $\mathcal{SVI}(\mathcal{X}, \mathcal{F}) \neq \emptyset$, then: monotone $\implies$ pseudomonotone $\implies$ Minty's condition

Note that while it has become common place to use the Minty condition in the analysis of VIs as it is much more general (see, for instance, He et al. (2022)), the Minty condition can at the cost of generality be in all

[2] For a definition of monotone, pseudomonotone and quasimonotone operators, Section 2.8.

the applications provided in this thesis replaced by the assumption that the VI $(\mathcal{X}, \mathcal{F})$ is quasimonotone with $\mathcal{X}$ non-empty, and compact by Lemma 3.1 and Proposition 3.1 of (He, 2017).

4.2 Algorithms for Variational Inequalities

We now turn our attention to the computation of solutions to variational inequalities. In what follows, for simplicity, we will restrict ourselves to VIs $(\mathcal{X}, \mathcal{F})$ in which $\mathcal{F}$ is singleton-valued, which we will for simplicity denote as $(\mathcal{X}, \boldsymbol{f})$. In future work, the algorithms and results provided in this chapter could be extended to the more general non-singleton-valued VI setting.

4.2.1 Computational Model

We will in this thesis consider two classes of methods to solve VIs, first-order and second-order methods which both belong to the class of kth order methods.

Definition 4.2.1 [kth-order methods]**.**

Given some $k \in \mathbb{N}_{++}$, a VI $(\mathcal{X}, \mathcal{F})$ for which the derivatives $\{\nabla^j \boldsymbol{f}\}_{j=1}^{k-1}$ are well defined, and an initial iterate $\boldsymbol{x}^{(0)} \in \mathcal{X}$, a **$k$th-order method $\boldsymbol{\mu}$** consists of an update function which generates the sequence of iterates $\{\boldsymbol{x}^{(t)}\}_t$ given for all $t = 0, 1, \dots$ by:

$$\boldsymbol{x}^{(t+1)} \doteq \boldsymbol{\mu}\left(\bigcup_{i=0}^{t}(\boldsymbol{x}^{(i)}, \{\nabla^j \boldsymbol{f}(\boldsymbol{x}^{(i)})\}_{j=0}^{k-1})\right)$$

The computational complexity results in this chapter will rely on the following computational model which has been broadly adopted by the literature (see, for instance, Cai et al. (2022)).

Definition 4.2.2 [VI Computational Model]**.**

Given a VI $(\mathcal{X}, \boldsymbol{f})$, and a kth-order method $\boldsymbol{\mu}$, the computational complexity of a is measured in term of the number of evaluations of the the functions $\boldsymbol{f}, \nabla \boldsymbol{f}, \dots, \nabla^k \boldsymbol{f}$.

Remark 4.2.1.

In line with the literature, the computational model we consider thus assumes that any other operation such as (Bregman) projection onto a set is a constant cost operation.

The computational results that exist in the literature, as well as the results we will present in this chapter hold in the following class of VIs.

Definition 4.2.3 [Lipschitz-Continuous VIs]**.**

Given a modulus of continuity $\lambda \geq 0$, a λ-**Lipschitz-continuous** VI is a VI $(\mathcal{X}, \boldsymbol{f})$ such that:

1. $\mathcal{X}$ is non-empty, compact, and convex
2. $\boldsymbol{f}$ is λ-Lipschitz continuous

4.2.2 Related works

Historically, the goal of the literature on solution methods for VIs has been to devise algorithms which are asymptotically guaranteed to converge to a strong or weak solution (Brezis and Brezis, 2011). An overwhelming majority of these works have focused on first-order methods for computing solutions of VIs, with higher order methods having been considered only in recent years (see, for instance, He et al. (2022); Huang and Zhang (2022)) While a strong solution of a VI is guaranteed to exist in continuous VIs, most results on the computational complexity of strong solutions, concerns the class of monotone VIs (see, for instance Cai et al. (2022)) with a few works focusing on VIs that satisfy the Minty condition (see, for instance, Diakonikolas (2020)).

The canonical algorithm to solve VIs is the projected gradient method (Cauchy et al., 1847; Nesterov, 1998) (also known under the names of the Subgradient method, Gradient Descent Ascent Method or Arrow-Hurwicz-Uzawa method (Arrow and Hurwicz, 1958; Arrow et al., 1958)). While asymptotic convergence of the projected gradient method to a solution can be shown for a subset of monotone VIs known as strongly monotone VIs[3], in general monotone VIs, only ergodic asymptotic convergence (i.e., asymptotic convergence of the averaged iterates) to a strong solution can be guaranteed. The earliest known algorithm with asymptotic convergence guarantees to a solution of a monotone VI, is the extragradient method, attributed to Korpelevich (1976). Following this earlier success, Popov (1980) introduced a closely related algorithm called the optimistic gradient method which he also showed to converge to a solution. These initial extragradient and optimistic gradient algorithms would eventually become much more sophisticated with a large body of work appearing on asymptotic convergence guarantees for variants of these earlier methods (e.g., (Solodov and Svaiter, 1999)). of the optimality operator

[3]Recall that for monotone VIs, the set of strong and weak solutions are equal, as such here "solution" refers to both strong and weak solutions.

More recently, the literature has turned its attention to algorithms with non-asymptotic guarantees, and in particular to ones that are guaranteed to compute a ε-strong or ε-weak solution of a VI in polynomial-time, i.e., in a number of evaluations of the optimality operator $\mathcal{F}$ which is polynomial in the inverse of the approximation parameter $1/\varepsilon$, the dimensionality n of the constraint set, and other relevant assumption specific parameters such as an upper bound on all of the values of the optimality operator. One of the earliest results in this direction was given by Nemirovski (2004), who introduced the conceptual mirror-Prox Method, an elegant generalization of the extragradient Method, and established that ε-strong and ε-weak solutions can be computed in $O(1/\varepsilon)$ operations by averaging the iterates of the algorithm under the assumption that the the VI is monotone, and the optimality operator is Lipschitz-continuous. Nemirovski's work was subsequently followed by a large body of work on more sophisticated algorithms (e.g., (Auslender and Teboulle, 2005; Diakonikolas, 2020)) for monotone VIs, and better computational results for the projection method (Gidel et al., 2018), the extragradient method (Gorbunov et al., 2022a; Golowich et al., 2020a; Cai et al., 2022) and the optimistic gradient method (Gorbunov et al., 2022b).

More recently, a number of works have considered first-order methods to compute a strong solution (e.g., Loizou et al. (2021); He et al. (2022); Diakonikolas (2020)) in VIs or a stationary point of the VI[4] (e.g., Liu et al. (2021)) that satisfy the Minty condition. The first-order methods considered by this more recent line of work on non-monotone variational inequalities include the extragradient method (e.g., (Wang and Ma, 2024; Ofem et al., 2023)), Tseng's method (e.g., (Censor et al., 2011; Thong et al., 2020; Uzor et al., 2023; Dung et al., 2024; Aremu et al., 2024)), and the optimistic gradient method (e.g., (Lin and Jordan, 2022)) and its variants.

[4]A (ε, δ) stationary point of a VI $(\mathcal{X}, \mathcal{F})$ is a point $\boldsymbol{x}^* \in \mathcal{X}$ s.t. for some $\delta \geq 0$ there exists $\boldsymbol{x} \in \mathcal{B}_\delta(\boldsymbol{x}^*)$ and $\boldsymbol{x}$ is a ε-strong solution. Convergence to this weaker solution concept is necessary for VIs in which $\mathcal{F}$ is not singleton-valued for technical reasons, and any future work that seeks to generalize the results in this section should adopt this weaker definition to prove their convergence results.

4.3 First-Order Methods

We will at present focus on first-order methods for VIs.

4.3.1 Mirror Gradient Algorithm

The canonical class of first-order methods to compute a strong solution for VIs is the class of **mirror gradient algorithms** (Algorithm 1, (Nemirovskij and Yudin, 1983)) which are parameterized by a kernel function $h : \mathcal{X} \to \mathbb{R}$ which induces a a Bregman divergence $\text{div}_h : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ that defines the update function $\boldsymbol{\mu}$ of the algorithm.

Definition 4.3.1 [Bregman Divergence]**.**
Given a set $\mathcal{X}$ and a **kernel function** $h : \mathcal{X} \to \mathbb{R}$, the **Bregman divergence** $\text{div}_h : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ associated with h is defined as:

$$\text{div}_h(\boldsymbol{x}, \boldsymbol{y}) \doteq h(\boldsymbol{x}) - h(\boldsymbol{y}) - \langle \nabla h(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle$$

We note the following properties of the Bregman divergence. For additional background, see, for instance Zhang and He (2018).

Remark 4.3.1 [Properties of the Bregman divergence]**.**
When h is convex, for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$, the Bregman divergence is positive, i.e., $\text{div}_h(\boldsymbol{x}, \boldsymbol{y}) \geq 0$. Further, if h is strictly convex, then $\text{div}_h(\boldsymbol{x}, \boldsymbol{y}) = 0$ iff $\boldsymbol{x} = \boldsymbol{y}$. In addition, if h is μ-strongly convex, then for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$, we have $\text{div}_h(\boldsymbol{x}, \boldsymbol{y}) \geq \frac{\mu}{2} \|\boldsymbol{x} - \boldsymbol{y}\|^2$.

Algorithm 1 Mirror Gradient Algorithm

Input: $\mathcal{X}, \boldsymbol{f}, h, \tau, \eta, \boldsymbol{x}^{(0)}$

Output: $\{\boldsymbol{x}^{(t)}\}_t$

1: **for** $t = 1, \ldots, \tau$ **do**

2: $\quad \boldsymbol{x}^{(t+1)} \leftarrow \arg\min_{\boldsymbol{x} \in \mathcal{X}} \left\{ \left\langle \boldsymbol{f}(\boldsymbol{x}^{(t)}), \boldsymbol{x} - \boldsymbol{x}^{(t)} \right\rangle + \frac{1}{2\eta} \text{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(t)}) \right\}$

return $\{\boldsymbol{x}^{(t)}\}_t$

When the kernel function is chosen s.t. $h(\boldsymbol{x}) \doteq \frac{1}{2}\|\boldsymbol{x}\|^2$, the Bregman divergence corresponds to the Euclidean square norm, i.e., $\text{div}_h(\boldsymbol{x}, \boldsymbol{y}) \doteq \|\boldsymbol{x} - \boldsymbol{y}\|^2$, in which case the mirror gradient method reduces to the well-known **projected gradient** method (Algorithm 2, (Cauchy et al., 1847)).

Algorithm 2 Project Gradient Algorithm

Input: $\mathcal{X}, \boldsymbol{f}, \tau, \eta, \boldsymbol{x}^{(0)}$

Output: $\{\boldsymbol{x}^{(t)}\}_t$

1: Initialize $\boldsymbol{x}^{(1)} \in \mathcal{X}$ arbitrarily

2: **for** $t = 1, \ldots, \tau$ **do**

3: $\quad \boldsymbol{x}^{(t+1)} \leftarrow \Pi_{\mathcal{X}}\left[\boldsymbol{x}^{(t)} - \eta \boldsymbol{f}(\boldsymbol{x}^{(t)})\right] = \arg\min_{\boldsymbol{x} \in \mathcal{X}} \left\{\left\langle \boldsymbol{f}(\boldsymbol{x}^{(t)}), \boldsymbol{x} - \boldsymbol{x}^{(t)} \right\rangle + {}^1\!/\!{}_{2\eta}\|\boldsymbol{x} - \boldsymbol{x}^{(t)}\|^2\right\}$

return $\{\boldsymbol{x}^{(t)}\}_t$

Unfortunately, while the average of the iterates of the mirror gradient method can be shown to converge to a strong solution asymptotically for monotone VIs with a Lipschitz-continuous optimality operator, it is in general only possible to prove polynomial-time computation of a ε-weak solution in such VIs which does not necessarily imply convergence to a ε-strong solution (see, for instance, Proposition 8 and Appendix D of Liu et al. (2021)). More importantly, in general the sequence of iterates generated by the mirror gradient method is not guaranteed to converge, as shown by the following example.

Example 4.3.1 [Non-Convergence of Gradient Method]**.**
Consider the VI $(\mathcal{X}, \boldsymbol{f})$ with $\mathcal{X} \doteq \mathbb{R}^2$ and $\boldsymbol{f}(x, y) = (-y, x)$. For this VI, we have $\mathcal{SVI}(\mathcal{X}, \boldsymbol{f}) = \mathcal{MVI}(\mathcal{X}, \boldsymbol{f}) = \{(0, 0)\}$. Suppose that $(\boldsymbol{x}^{(0)}, \boldsymbol{y}^{(0)}) \neq (0, 0)$, then for any $\eta > 0$ the iterates generated by the gradient method are given by:

$$(x^{(t)}, y^{(t)}) \doteq \left(x^{(0)} - \eta \sum_{k=1}^{t} y^{(k-1)}, y^{(0)} + \eta \sum_{k=1}^{t} x^{(k-1)}\right) \qquad \forall t \in \mathbb{N}_{++} \tag{4.6}$$

and as such are unbounded, i.e., $\|(\boldsymbol{x}^{(t)}, \boldsymbol{y}^{(t)})\| \to \infty$.

4.3.2 Mirror Extragradient Algorithm

As the iterates of the mirror gradient method do not asymptotically converge to a strong or weak solution, and it is not possible to obtain polynomial-time computation of a ε-strong solution by averaging the iterates, we now introduce a novel class of first order methods, namely the class of **mirror extragradient algorithms** which similar to the class of mirror gradient methods are parameterized by a kernel function $h : \mathcal{X} \to \mathbb{R}$ that induces a Bregman divergence $\text{div}_h : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ defining the update function $\boldsymbol{\mu}$ of the algorithm.

Algorithm 3 Mirror Extragradient Algorithm

Input: $\mathcal{X}, \boldsymbol{f}, h, \tau, \eta, \boldsymbol{x}^{(0)}$

Output: $\{\boldsymbol{x}^{(t+0.5)}, \boldsymbol{x}^{(t+1)}\}_t$

1: **for** $t = 1, \dots, \tau$ **do**

2: $\quad \boldsymbol{x}^{(t+0.5)} \leftarrow \arg\min_{\boldsymbol{x} \in \mathcal{X}} \left\{ \left\langle \boldsymbol{f}(\boldsymbol{x}^{(t)}), \boldsymbol{x} - \boldsymbol{x}^{(t)} \right\rangle + \frac{1}{2\eta} \text{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(t)}) \right\}$

3: $\quad \boldsymbol{x}^{(t+1)} \leftarrow \arg\min_{\boldsymbol{x} \in \mathcal{X}} \left\{ \left\langle \boldsymbol{f}(\boldsymbol{x}^{(t+0.5)}), \boldsymbol{x} - \boldsymbol{x}^{(t)} \right\rangle + \frac{1}{2\eta} \text{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(t)}) \right\}$

return $\{\boldsymbol{x}^{(t+0.5)}, \boldsymbol{x}^{(t+1)}\}_t$

The mirror extragradient algorithm (Algorithm 3) generalizes the well-known extragradient algorithm which is known to asymptotically converge to a strong solution (Popov, 1980), and allows for the polynomial-time computation of a ε-strong solution (Nemirovski, 2004; Golowich et al., 2020b; Cai et al., 2022). In particular, when the kernel function for the mirror extragradient method is chosen s.t. $h(\boldsymbol{x}) \doteq \frac{1}{2} \|\boldsymbol{x}\|^2$, the Bregman divergence corresponds to the Euclidean square norm, i.e., $\text{div}_h(\boldsymbol{x}, \boldsymbol{y}) \doteq \|\boldsymbol{x} - \boldsymbol{y}\|^2$, in which case the mirror gradient method reduces to the extragradient method (Algorithm 4).

Algorithm 4 Extragradient Algorithm

Input: $\mathcal{X}, \boldsymbol{f}, \tau, \eta, \boldsymbol{x}^{(0)}$

Output: $\{\boldsymbol{x}^{(t+0.5)}, \boldsymbol{x}^{(t+1)}\}_t$

1: **for** $t = 1, \dots, \tau$ **do**

2: $\quad \boldsymbol{x}^{(t+0.5)} \leftarrow \Pi_{\mathcal{X}} \left[\boldsymbol{x}^{(t)} - \eta \boldsymbol{f}(\boldsymbol{x}^{(t)})\right] = \arg\min_{\boldsymbol{x} \in \mathcal{X}} \left\{ \left\langle \boldsymbol{f}(\boldsymbol{x}^{(t)}), \boldsymbol{x} - \boldsymbol{x}^{(t)} \right\rangle + {}^1\!/_{2\eta} \|\boldsymbol{x} - \boldsymbol{x}^{(t)}\|^2 \right\}$

3: $\quad \boldsymbol{x}^{(t+1)} \leftarrow \Pi_{\mathcal{X}} \left[\boldsymbol{x}^{(t)} - \eta \boldsymbol{f}(\boldsymbol{x}^{(t+0.5)})\right] = \arg\min_{\boldsymbol{x} \in \mathcal{X}} \left\{ \left\langle \boldsymbol{f}(\boldsymbol{x}^{(t+0.5)}), \boldsymbol{x} - \boldsymbol{x}^{(t)} \right\rangle + {}^1\!/_{2\eta} \|\boldsymbol{x} - \boldsymbol{x}^{(t)}\|^2 \right\}$

return $\{\boldsymbol{x}^{(t+0.5)}, \boldsymbol{x}^{(t+1)}\}_t$

A seminal result by Nemirovski (2004) shows that the average of the iterates output by the extragradient algorithm are a ε-strong solution for any monotone VI with a Lipschitz-continuous optimality operator when the algorithm is run for $\tau \in O({}^1\!/_\varepsilon)$ time-steps. Additionally, Golowich et al. (2020b); Cai et al. (2022) show that in the same setting, the tth iterate is a ε-strong solution for $\tau \in O({}^1\!/_{\varepsilon^2})$ time-steps. More recently, Huang and Zhang (2023) have extended these polynomial-time computation result to VIs which satisfy the weaker Minty condition rather than monotonicity assumption, showing that there exists some $k \leq \tau$ s.t. the kth iterate of the extragradient algorithm is a ε-strong solution for $\tau \in O(1/\varepsilon^2)$.[5] We extend at present this result to mirror extragradient algorithm. In order to prove, our result we prove several lemmas. We start with a technical lemma on Bregman divergences.

Lemma 4.3.1 [Bregman Triangle Lemma]**.**

Consider the Bregman divergence $\mathrm{div}_h : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$ associated with a differentiable kernel function $h : \mathcal{X} \to \mathbb{R}$. Let $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{X}$, we then have:

$$\mathrm{div}_h(\boldsymbol{x}, \boldsymbol{z}) + \mathrm{div}_h(\boldsymbol{y}, \boldsymbol{x}) - \mathrm{div}_h(\boldsymbol{y}, \boldsymbol{z}) = \langle \nabla h(\boldsymbol{x}) - \nabla h(\boldsymbol{z}), \boldsymbol{x} - \boldsymbol{y} \rangle. \tag{4.7}$$

Proof of Lemma 4.3.1

For all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{X}$, we have:

$$\begin{aligned} &\mathrm{div}_h(\boldsymbol{x}, \boldsymbol{z}) + \mathrm{div}_h(\boldsymbol{y}, \boldsymbol{x}) - \mathrm{div}_h(\boldsymbol{y}, \boldsymbol{z}) \\ &= [h(\boldsymbol{x}) - h(\boldsymbol{z}) - \langle \nabla h(\boldsymbol{z}), \boldsymbol{x} - \boldsymbol{z} \rangle] + [h(\boldsymbol{y}) - h(\boldsymbol{x}) - \langle \nabla h(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle] - [h(\boldsymbol{y}) - h(\boldsymbol{z}) - \langle \nabla h(\boldsymbol{z}), \boldsymbol{y} - \boldsymbol{z} \rangle] \\ &= -\langle \nabla h(\boldsymbol{z}), \boldsymbol{x} - \boldsymbol{z} \rangle - \langle \nabla h(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle + \langle \nabla h(\boldsymbol{z}), \boldsymbol{y} - \boldsymbol{z} \rangle \\ &= \langle \nabla h(\boldsymbol{z}) - \nabla h(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle. \end{aligned}$$

With the above technical lemma in hand, we are now ready to prove a progress lemma for the mirror extragradient method, which describes how the algorithm progresses from one iteration to another. Note that under the Minty condition, the following lemma implies convergence to a weak solution since setting $\boldsymbol{x} \doteq \boldsymbol{x}^* \in \mathcal{MVI}(\mathcal{X}, \boldsymbol{f})$, we obtain $\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(k)}) > \mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(k+1)})$ for all $k \in \mathbb{N}$ (i.e., the distance to the weak solution $\boldsymbol{x}^*$ is strictly decreasing). Below, we first introduce a condition necessary to hold for the result of the lemma to hold, and then state the lemma.

Definition 4.3.2 [Pathwise Bregman-Continuity]**.**

A VI $(\mathcal{X}, \boldsymbol{f})$ is **pathwise Bregman continuous** over the outputs of the mirror extragradient method $\{\boldsymbol{x}^{(t+0.5)}, \boldsymbol{x}^{(t)}\}_t$, i.e., there exists $\lambda \geq 0$, s.t. for all $t \in [\tau]$, $\frac{1}{2}\|\boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}) - \boldsymbol{f}(\boldsymbol{x}^{(k)})\|^2 \leq \lambda^2 \mathrm{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})$.

[5] This type of convergence is known as a best-iterate convergence.

Lemma 4.3.2 [Mirror Extragradient Progress]**.**

Consider the mirror extragradient algorithm (Algorithm 3), run with a VI $(\mathcal{X}, \boldsymbol{f})$, and a 1-strongly-convex kernel function h, a step size $\eta > 0$, a time horizon $\tau \in \mathbb{N}$, and outputs $\{\boldsymbol{x}^{(t+0.5)}, \boldsymbol{x}^{(t+1)}\}_t$. Suppose that $(\mathcal{X}, \boldsymbol{f})$ is pathwise Bregman-continuous, i.e., there exists $\lambda \geq 0$, s.t. $\frac{1}{2}\|\boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}) - \boldsymbol{f}(\boldsymbol{x}^{(k)})\|^2 \leq \lambda^2 \mathrm{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})$. Then, for all $k \in \mathbb{N}$ and $\boldsymbol{x} \in \mathcal{X}$, the following inequality holds for its outputs $\{\boldsymbol{x}^{(t+0.5)}, \boldsymbol{x}^{(t+1)}\}_t$:

$$\mathrm{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(k)}) - \mathrm{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(k+1)}) \geq \eta\langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}\rangle + \left(1 - (\eta\lambda)^2\right) \mathrm{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) \tag{4.8}$$

Proof of Lemma 4.3.2

By the first order optimality conditions of $\boldsymbol{x}^{(k+0.5)}$, we have for all $\boldsymbol{x} \in \mathcal{X}$:

$$\langle \boldsymbol{f}(\boldsymbol{x}^{(k)}) + \frac{1}{\eta}\langle \nabla h(\boldsymbol{x}^{(k+0.5)}) - \nabla h(\boldsymbol{x}^{(k)}), \boldsymbol{x} - \boldsymbol{x}^{(k+0.5)}\rangle \geq 0.$$

Substituting $\boldsymbol{x} = \boldsymbol{x}^{(k+1)}$ above, we have:

$$\begin{aligned}\langle \boldsymbol{f}(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k+0.5)}\rangle &\geq \frac{1}{\eta}\langle \nabla h(\boldsymbol{x}^{(k)}) - \nabla h(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k+0.5)}\rangle \\ &= \frac{1}{\eta}\left(\mathrm{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) + \mathrm{div}_h(\boldsymbol{x}^{(k+1)}, \boldsymbol{x}^{(k+0.5)}) - \mathrm{div}_h(\boldsymbol{x}^{(k+1)}, \boldsymbol{x}^{(k)})\right).\end{aligned} \tag{4.9}$$

where the last line was obtained by Lemma 4.3.1.

On the other hand, by the optimality condition at $\boldsymbol{x}^{(k+1)}$, we have for all $\boldsymbol{x} \in \mathcal{X}$:

$$\langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x} - \boldsymbol{x}^{(k+1)}\rangle + \frac{1}{\eta}\langle \nabla h(\boldsymbol{x}^{(k+1)}) - \nabla h(\boldsymbol{x}^{(k)}), \boldsymbol{x} - \boldsymbol{x}^{(k+1)}\rangle \geq 0 \ .$$

Hence, for all $\boldsymbol{x} \in \mathcal{X}$:

$$\begin{aligned}\langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x} - \boldsymbol{x}^{(k+1)}\rangle &\geq \frac{1}{\eta}\langle \nabla h(\boldsymbol{x}^{(k)}) - \nabla h(\boldsymbol{x}^{(k+1)}), \boldsymbol{x} - \boldsymbol{x}^{(k+1)}\rangle \\ &= \frac{1}{\eta}\left(\mathrm{div}_h(\boldsymbol{x}^{(k+1)}, \boldsymbol{x}^{(k)}) + \mathrm{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(k+1)}) - \mathrm{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(k)})\right) \ .\end{aligned}$$

where the last line was once again obtained by Lemma 4.3.1.

Continuing with the above inequality, for any given $\boldsymbol{x} \in \mathcal{X}$, we have:

$$\frac{1}{\eta}\left(\operatorname{div}_h(\boldsymbol{x}^{(k+1)}, \boldsymbol{x}^{(k)}) + \operatorname{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(k+1)}) - \operatorname{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(k)})\right) \tag{4.10}$$

$$\leq \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x} - \boldsymbol{x}^{(k+1)} \rangle \tag{4.11}$$

$$= \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x} - \boldsymbol{x}^{(k+0.5)} \rangle + \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k+1)} \rangle \tag{4.12}$$

$$= \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x} - \boldsymbol{x}^{(k+0.5)} \rangle + \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}) - \boldsymbol{f}(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k+1)} \rangle$$

$$+ \langle \boldsymbol{f}(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k+1)} \rangle \tag{4.13}$$

$$\leq \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x} - \boldsymbol{x}^{(k+0.5)} \rangle + \|\boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}) - \boldsymbol{f}(\boldsymbol{x}^{(k)})\| \cdot \|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k+1)}\|$$

$$+ \langle \boldsymbol{f}(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k+1)} \rangle \tag{4.14}$$

where the final line follows by the Cauchy-Schwarz inequality (Cauchy, 1821; Schwarz, 1884).

Recall that by the arithmetic mean-geometric mean inequality, $\forall x, y \in \mathbb{R}_+$, $\frac{x+y}{2} \geq \sqrt{xy}$. Hence, applying the inequality with $x = \eta \|\boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}) - \boldsymbol{f}(\boldsymbol{x}^{(k)})\|^2$ and $y = {}^1/_\eta \|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k+1)}\|^2$

$$\frac{1}{\eta}\left(\operatorname{div}_h(\boldsymbol{x}^{(k+1)}, \boldsymbol{x}^{(k)}) + \operatorname{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(k+1)}) - \operatorname{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(k)})\right) \tag{4.15}$$

$$\leq \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x} - \boldsymbol{x}^{(k+0.5)} \rangle + \frac{\eta \|\boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}) - \boldsymbol{f}(\boldsymbol{x}^{(k)})\|^2}{2}$$

$$+ \frac{\|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k+1)}\|^2}{2\eta} + \langle \boldsymbol{f}(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k+1)} \rangle \tag{4.16}$$

$$\leq \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x} - \boldsymbol{x}^{(k+0.5)} \rangle + \eta \lambda^2 \operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})$$

$$+ \frac{\|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k+1)}\|^2}{2\eta} + \langle \boldsymbol{f}(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k+1)} \rangle \tag{4.17}$$

where the last line was obtained by the assumption that there exists $\lambda \geq 0$, s.t. $\frac{1}{2}\|\boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}) - \boldsymbol{f}(\boldsymbol{x}^{(k)})\|^2 \leq \lambda^2 \operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})$.

Additionally, note that by strong convexity of h, we have $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$, $\operatorname{div}_h(\boldsymbol{x}, \boldsymbol{y}) \geq {}^1/_2 \|\boldsymbol{x} - \boldsymbol{y}\|^2$. Hence, continuing we have:

$$\frac{1}{\eta}\left(\operatorname{div}_h(\boldsymbol{x}^{(k+1)}, \boldsymbol{x}^{(k)}) + \operatorname{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(k+1)}) - \operatorname{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(k)})\right) \tag{4.18}$$

$$\leq \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x} - \boldsymbol{x}^{(k+0.5)} \rangle + \eta \lambda^2 \operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})$$

$$+ \frac{\operatorname{div}_h(\boldsymbol{x}^{(k+1)}, \boldsymbol{x}^{(k+0.5)})}{\eta} + \langle \boldsymbol{f}(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k+1)} \rangle. \tag{4.19}$$

Plugging Equation (4.9) into the above, we have:

$$\frac{1}{\eta}\left(\mathrm{div}_h(\boldsymbol{x}^{(k+1)},\boldsymbol{x}^{(k)})+\mathrm{div}_h(\boldsymbol{x},\boldsymbol{x}^{(k+1)})-\mathrm{div}_h(\boldsymbol{x},\boldsymbol{x}^{(k)})\right)$$
$$\leq \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}),\boldsymbol{x}-\boldsymbol{x}^{(k+0.5)}\rangle+\eta\lambda^2\mathrm{div}_h(\boldsymbol{x}^{(k+0.5)},\boldsymbol{x}^{(k)})$$
$$+\frac{\mathrm{div}_h(\boldsymbol{x}^{(k+1)},\boldsymbol{x}^{(k+0.5)})}{\eta}-\frac{1}{\eta}\left(\mathrm{div}_h(\boldsymbol{x}^{(k+0.5)},\boldsymbol{x}^{(k)})+\mathrm{div}_h(\boldsymbol{x}^{(k+1)},\boldsymbol{x}^{(k+0.5)})-\mathrm{div}_h(\boldsymbol{x}^{(k+1)},\boldsymbol{x}^{(k)})\right)$$
$$\leq \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}),\boldsymbol{x}-\boldsymbol{x}^{(k+0.5)}\rangle+\left(\eta\lambda^2-\frac{1}{\eta}\right)\mathrm{div}_h(\boldsymbol{x}^{(k+0.5)},\boldsymbol{x}^{(k)})+\frac{1}{\eta}\mathrm{div}_h(\boldsymbol{x}^{(k+1)},\boldsymbol{x}^{(k)})$$

Canceling out terms, we simplify the above inequality into:

$$\frac{1}{\eta}\left(\mathrm{div}_h(\boldsymbol{x},\boldsymbol{x}^{(k+1)})-\mathrm{div}_h(\boldsymbol{x},\boldsymbol{x}^{(k)})\right)\leq \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}),\boldsymbol{x}-\boldsymbol{x}^{(k+0.5)}\rangle+\left(\eta\lambda^2-\frac{1}{\eta}\right)\mathrm{div}_h(\boldsymbol{x}^{(k+0.5)},\boldsymbol{x}^{(k)}) \tag{4.20}$$

Multiplying both sides by $-\eta<0$, we obtain the lemma statement.

While the above lemma implies asymptotic convergence to a strong solution, to show polynomial-time computation of a ε-strong solution, we have to bound the progress of the intermediate iterates $\mathrm{div}_h(\boldsymbol{x}^{(k+0.5)},\boldsymbol{x}^{(k)})$ as a function of the time horizon algorithm. In the proof of the following theorem we show that we can bound this quantity, assuming that the kernel function is in addition κ-Lipschitz-smooth.

Theorem 4.3.1 [Mirror Extragradient Method Convergence]**.**
Let $(\mathcal{X},\boldsymbol{f})$ be a VI satisfying the Minty condition, and h a 1-strongly-convex and κ-Lipschitz-smooth kernel function. Consider the mirror extragradient algorithm (Algorithm 3) run with the VI $(\mathcal{X},\boldsymbol{f})$, the kernel function h, a step size $\eta>0$, a time horizon $\tau\in\mathbb{N}$, and outputs $\{\boldsymbol{x}^{(t+0.5)},\boldsymbol{x}^{(t+1)}\}_t$. Suppose that there exists $\lambda\in(0,\frac{1}{\sqrt{2}\eta}$, s.t. $\frac{1}{2}\|\boldsymbol{f}(\boldsymbol{x}^{(k+0.5)})-\boldsymbol{f}(\boldsymbol{x}^{(k)})\|^2\leq\lambda^2\mathrm{div}_h(\boldsymbol{x}^{(k+0.5)},\boldsymbol{x}^{(k)})$. Then, the following bound holds:

$$\min_{k=0,\dots,\tau}\max_{\boldsymbol{x}\in\mathcal{X}}\langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}),\boldsymbol{x}^{(k+0.5)}-\boldsymbol{x}\rangle\leq\frac{2(1+\kappa)\mathrm{diam}(\mathcal{X})}{\eta}\frac{\sqrt{\mathrm{div}_h(\boldsymbol{x}^*,\boldsymbol{x}^{(0)})}}{\sqrt{\tau}}$$

where $\boldsymbol{x}^*\in\mathcal{MVI}(\mathcal{X},\boldsymbol{f})$ is a weak solution of the VI $(\mathcal{X},\boldsymbol{f})$.

In addition, let $\boldsymbol{x}^{(\tau)}_{\text{best}}\in\operatorname*{arg\,min}_{\boldsymbol{x}^{(k+0.5)}:k=0,\dots,\tau}\mathrm{div}_h(\boldsymbol{x}^{(k+0.5)},\boldsymbol{x}^{(k)})$, then, for some choice of time horizon $\tau\in O(\frac{\kappa^2\mathrm{diam}(\mathcal{X})^2\mathrm{div}_h(\boldsymbol{x}^*,\boldsymbol{x}^{(0)})}{\eta^2\varepsilon^2})$, $\boldsymbol{x}^{(\tau)}_{\text{best}}$ is a ε-strong solution of $(\mathcal{X},\boldsymbol{f})$, and $\lim_{t\to\infty}\boldsymbol{x}^{(t+0.5)}=\lim_{t\to\infty}\boldsymbol{x}^{(t)}\in\mathcal{SVI}(\mathcal{X},\boldsymbol{f})$ is a strong solution of the VI $(\mathcal{X},\boldsymbol{f})$.

Proof of Theorem 4.3.1

Taking Lemma 4.3.2 with $\boldsymbol{x} \doteq \boldsymbol{x}^* \in \mathcal{MVI}(\mathcal{X}, \boldsymbol{f})$, we have:

$$\begin{aligned}\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(k)}) - \operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(k+1)}) &\geq \eta \underbrace{\langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^* \rangle}_{\geq 0} + \left(1 - (\eta\lambda)^2\right) \operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) \\ &\geq \left(1 - (\eta\lambda)^2\right) \operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})\end{aligned}$$

Multiplying both sides by $\left(1 - (\eta\lambda)^2\right)^{-1} > 0$, we have:

$$\operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) \leq \frac{1}{1 - (\eta\lambda)^2} \left(\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(k)}) - \operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(k+1)})\right) \tag{4.21}$$

Summing up for $k = 0, \ldots, \tau$:

$$\begin{aligned}\sum_{k=0}^{\tau} \operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) &\leq \frac{1}{1 - (\eta\lambda)^2} \sum_{k=0}^{\tau} \left(\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(k)}) - \operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(k+1)})\right) &(4.22)\\ &\leq \frac{1}{1 - (\eta\lambda)^2} \left(\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)}) - \operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(\tau+1)})\right) &(4.23)\\ &\leq \frac{1}{1 - (\eta\lambda)^2} \operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)}) &(4.24)\end{aligned}$$

Dividing both sides by τ, we have:

$$\begin{aligned}\frac{1}{\tau} \sum_{k=0}^{\tau} \operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) &\leq \frac{1}{\tau(1 - (\eta\lambda)^2)} \left(\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})\right) &(4.25)\\ \min_{k=0,\ldots,\tau} \operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) &\leq \frac{1}{\tau(1 - (\eta\lambda)^2)} \left(\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})\right) &(4.26)\end{aligned}$$

We can transform this convergence into a convergence in terms of the primal gap function. Now, recall by the first order optimality conditions of $\boldsymbol{x}^{(k+0.5)}$, we have for all $\boldsymbol{x} \in \mathcal{X}$:

$$\langle \boldsymbol{f}(\boldsymbol{x}^{(k)}) + \frac{1}{\eta} \langle \nabla h(\boldsymbol{x}^{(k+0.5)}) - \nabla h(\boldsymbol{x}^{(k)}), \boldsymbol{x} - \boldsymbol{x}^{(k+0.5)} \rangle \geq 0.$$

Re-organizing, for all $\boldsymbol{x} \in \mathcal{X}$, and $k \in \mathbb{N}$ we have:

$$\begin{aligned}\langle \boldsymbol{f}(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x} \rangle &\leq \frac{1}{\eta} \left\| \nabla h(\boldsymbol{x}^{(k+0.5)}) - \nabla h(\boldsymbol{x}^{(k)}) \right\| \left\| \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x} \right\| &(4.27)\\ &\leq \frac{\operatorname{diam}(\mathcal{X})}{\eta} \left\| \nabla h(\boldsymbol{x}^{(k+0.5)}) - \nabla h(\boldsymbol{x}^{(k)}) \right\| &(4.28)\\ &\leq \frac{\operatorname{diam}(\mathcal{X})\kappa}{\eta} \left\| \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k)} \right\| &(4.29)\end{aligned}$$

where the last line follow from h being κ-Lipschitz-smooth.

Now, with the above inequality in hand, notice that for all $\boldsymbol{x} \in \mathcal{X}$ and $k \in \mathbb{N}$, we have:

$$\begin{aligned}
\langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x} \rangle &= \langle \boldsymbol{f}(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x} \rangle + \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}) - \boldsymbol{f}(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x} \rangle \\
&\leq \frac{\operatorname{diam}(\mathcal{X})\kappa}{\eta} \|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k)}\| + \|\boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}) - \boldsymbol{f}(\boldsymbol{x}^{(k)})\| \cdot \|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}\| \\
&\leq \frac{\operatorname{diam}(\mathcal{X})\kappa}{\eta} \|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k)}\| + \lambda \sqrt{2\operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})} \cdot \|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}\| \\
&\leq \operatorname{diam}(\mathcal{X}) \left(\frac{\kappa}{\eta} + \lambda\right) \sqrt{2\operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})}
\end{aligned}$$

where the penultimate line was obtained by the assumption that there exists $\lambda \geq 0$, s.t. $\frac{1}{2}\|\boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}) - \boldsymbol{f}(\boldsymbol{x}^{(k)})\|^2 \leq \lambda^2 \operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})$, and the last line from the strong convexity of h, which means that we have $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$, $\operatorname{div}_h(\boldsymbol{x}, \boldsymbol{y}) \geq 1/2\|\boldsymbol{x} - \boldsymbol{y}\|^2$.

Now, let $k^* \in \arg\min_{k=0,\dots,\tau} \operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})$, we then have for all $\boldsymbol{x} \in \mathcal{X}$:

$$\begin{aligned}
\langle \boldsymbol{f}(\boldsymbol{x}^{(k^*+0.5)}), \boldsymbol{x}^{(k^*+0.5)} - \boldsymbol{x} \rangle &\leq \operatorname{diam}(\mathcal{X}) \left(\frac{\kappa}{\eta} + \lambda\right) \sqrt{2\operatorname{div}_h(\boldsymbol{x}^{(k^*+0.5)}, \boldsymbol{x}^{(k^*)})} \\
&= \operatorname{diam}(\mathcal{X}) \left(\frac{\kappa}{\eta} + \lambda\right) \sqrt{2 \min_{k=0,\dots,\tau} \operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})}
\end{aligned}$$

Now, plugging Equation (4.26) in the above, we have for all $\boldsymbol{x} \in \mathcal{X}$:

$$\langle \boldsymbol{f}(\boldsymbol{x}^{(k^*+0.5)}), \boldsymbol{x}^{(k^*+0.5)} - \boldsymbol{x} \rangle \leq \frac{\sqrt{2}\operatorname{diam}(\mathcal{X}) \left(\frac{\kappa}{\eta} + \lambda\right)}{\sqrt{1 - (\eta\lambda)^2}} \frac{\sqrt{\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})}}{\sqrt{\tau}}$$

Now, by the assumption that $\eta \leq \frac{1}{\sqrt{2}\lambda} \leq \frac{1}{\lambda}$, we have:

$$\begin{aligned}
\langle \boldsymbol{f}(\boldsymbol{x}^{(k^*+0.5)}), \boldsymbol{x}^{(k^*+0.5)} - \boldsymbol{x} \rangle &\leq \frac{\sqrt{2}\operatorname{diam}(\mathcal{X}) \left(\frac{\kappa}{\eta} + \frac{1}{\eta}\right)}{\sqrt{1 - (\eta\lambda)^2}} \frac{\sqrt{\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})}}{\sqrt{\tau}} \\
&= \frac{(1+\kappa)\sqrt{2}\operatorname{diam}(\mathcal{X})}{\eta\sqrt{1 - (\eta\lambda)^2}} \frac{\sqrt{\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})}}{\sqrt{\tau}} \\
&= \frac{(1+\kappa)\sqrt{2}\operatorname{diam}(\mathcal{X})}{\eta\sqrt{1 - (1/\sqrt{2})^2}} \frac{\sqrt{\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})}}{\sqrt{\tau}} \\
&= \frac{2(1+\kappa)\operatorname{diam}(\mathcal{X})}{\eta} \frac{\sqrt{\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})}}{\sqrt{\tau}} \\
&= \frac{2(1+\kappa)\operatorname{diam}(\mathcal{X})}{\eta} \frac{\sqrt{\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})}}{\sqrt{\tau}}
\end{aligned}$$

That is, we have:

$$\min_{k=0,\dots,\tau} \max_{\boldsymbol{x}\in\mathcal{X}} \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}\rangle \leq \max_{\boldsymbol{x}\in\mathcal{X}} \langle \boldsymbol{f}(\boldsymbol{x}^{(k^*+0.5)}), \boldsymbol{x}^{(k^*+0.5)} - \boldsymbol{x}\rangle$$
$$\leq \frac{2(1+\kappa)\mathrm{diam}(\mathcal{X})}{\eta} \frac{\sqrt{\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})}}{\sqrt{\tau}}$$

In addition, for any $\varepsilon \geq 0$, letting $\frac{2(1+\kappa)\mathrm{diam}(\mathcal{X})}{\eta} \frac{\sqrt{\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})}}{\sqrt{\tau}} \leq \varepsilon$, and solving for τ, we have:

$$\frac{2(1+\kappa)\mathrm{diam}(\mathcal{X})}{\eta} \frac{\sqrt{\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})}}{\sqrt{\tau}} \leq \varepsilon$$
$$\frac{4(1+\kappa)^2\mathrm{diam}(\mathcal{X})^2}{\eta^2} \frac{\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})}{\varepsilon^2} \leq \tau$$

That is, $\boldsymbol{x}^{(\tau)}_{\mathrm{best}} \in \arg\min_{\boldsymbol{x}^{(k+0.5)}:k=0,\dots,\tau} \mathrm{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})$ is a ε-strong solution after $\frac{4(1+\kappa)^2\mathrm{diam}(\mathcal{X})^2}{\eta^2} \frac{\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})}{\varepsilon^2}$ iterations of the mirror extragradient algorithm.

Finally, notice that we have $\lim_{k\to\infty} \max_{\boldsymbol{x}\in\mathcal{X}} \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}\rangle = \lim_{\tau\to\infty} \min_{k=0,\dots,\tau} \max_{\boldsymbol{x}\in\mathcal{X}} \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}\rangle = 0$ and $\lim_{k\to\infty} \mathrm{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) = \lim_{\tau\to\infty} \min_{k=0,\dots,\tau} \mathrm{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) = 0$. Hence, $\lim_{t\to\infty} \boldsymbol{x}^{(t+0.5)} = \lim_{t\to\infty} \boldsymbol{x}^{(t)} = \boldsymbol{x}^{**}$ is a strong solution of the VI $(\mathcal{X}, \boldsymbol{f})$.

With the main theorem of this section proven, some remarks are in order.

Remark 4.3.2.

First, note that the assumption that h is 1-strongly-convex is without loss of generality since any μ-strongly-convex kernel h' can be converted to a 1-strongly-convex kernel $\frac{1}{\mu}h'$.

Second, while for ease of exposition we assume that the VI is λ-Lipschitz-continuous, this assumption can more generally be weakened to 1) the VI is continuous, and 2) for all $t \in \mathbb{N}$, there exists $\lambda \geq 0$ s.t. $\mathrm{div}_h(f(\boldsymbol{x}^{(t+0.5)}), f(\boldsymbol{x}^{(t)})) \leq \lambda \mathrm{div}_h(\boldsymbol{x}^{(t+0.5)}, \boldsymbol{x}^{(t)})$. The second part of this assumption can be interpreted as Lipschitz-continuity w.r.t. the Bregman divergence div_h over the trajectories of the mirror extragradient algorithm. As we will show in Chapter 6, such an assumption can be useful when the optimality operator might not satisfy Lipschitz-continuity.

4.3.3 Local Convergence of Mirror Extragradient

Unfortunately, beyond VIs for which the Minty condition holds, it seems implausible to devise a first-order method that converges to strong solutions. To see this, observe the following example.

Example 4.3.2 [Non-convergence in the absence of the Minty condition]**.**
Consider the VI $(\mathcal{X}, f)$ where $\mathcal{X} \doteq \mathbb{R}$ and $f(x) \doteq 1 - x^2$. The set of strong solutions of VI is given by $\mathcal{SVI}(\mathcal{X}, f) = \{-1, 1\}$. Notice that for any any $x > 1$, $f(x) < 0$. As such, for the mirror (extra)gradient method, for any choice of step size $\eta > 0$, if the initial iterate is initialized s.t. $x^{(0)} > 1$, then $x^{(0)} \to \infty$.

This is perhaps not surprising, since the computation of a ε-strong solution for Lipschitz-continuous VIs is in general a PPAD-complete problem (Kapron and Samieefar, 2024). Nevertheless, in the above example one can see that for $x^{(0)} < 1$, the mirror (extra)gradient algorithm converges to the strong solution $x^* = -1$. This then begs the question—under what conditions can one guarantee the local convergence of the mirror extragradient algorithm to a strong solution. As we will show, we can guarantee local convergence by assuming a local variant of Minty's condition. To define this condition, we have to first define the notion of local weak and local strong solutions, recently introduced by Aussel and Chaipunya (2024).

Definition 4.3.3 [Local Weak and Strong Solution]**.**
Consider a VI $(\mathcal{X}, \mathcal{F})$ be a VI. Let $\delta \geq 0$ be a **locality parameter**.

A δ**-local strong solution** of the VI is a $\boldsymbol{x}^* \in \mathcal{X}$ that satisfies:

$$\exists \boldsymbol{f}(\boldsymbol{x}^*) \in \mathcal{F}(\boldsymbol{x}^*), \qquad \max_{\boldsymbol{x} \in \mathcal{X} \cap \mathcal{B}_\delta(\boldsymbol{x}^*)} \langle \boldsymbol{f}(\boldsymbol{x}^*), \boldsymbol{x} - \boldsymbol{x}^* \rangle \geq 0 \tag{4.30}$$

A δ**-local weak solution** of the VI is a $\boldsymbol{x}^* \in \mathcal{X}$ that satisfies:

$$\exists \boldsymbol{f}(\boldsymbol{x}^*) \in \mathcal{F}(\boldsymbol{x}), \qquad \max_{\boldsymbol{x} \in \mathcal{X} \cap \mathcal{B}_\delta(\boldsymbol{x}^*)} \langle \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x} - \boldsymbol{x}^* \rangle \leq 0 \tag{4.31}$$

We denote the set of δ-local strong (resp. weak) solutions of a VI $(\mathcal{X}, \mathcal{F})$ by $\mathcal{LSVI}^\delta(\mathcal{X}, \mathcal{F})$ (resp. $\mathcal{LMVI}^\delta(\mathcal{X}, \mathcal{F})$).

We note that for any VI $(\mathcal{X}, \mathcal{F})$ with $\mathcal{X}$ convex, local strong solutions are not of great interest since they coincide with (global) strong solutions (see section 3.2. of Aussel and Chaipunya (2024). Note that this observation also suggests that the computation of a δ-local strong solution is also PPAD-complete in Lipschitz-continuous VIs.

Nevertheless, as we will show next, local weak solutions can be of great interest to show local convergence to a strong solution. To understand how the above condition can allow us convergence, recall by Lemma 4.3.2 the iterates of the mirror extragradient algorithm satisfy the following for all $t \in \mathbb{N}$:

$$\operatorname{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(k)}) - \operatorname{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(k+1)}) \geq \eta \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x} \rangle + \left(1 - (\eta\lambda)^2\right) \operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})$$

Suppose that the kernel function h is strictly convex, and that the algorithm has not yet converged, i.e., $\boldsymbol{x}^{(k+0.5)} \neq \boldsymbol{x}^{(k)}$, then $\operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) > 0$, and we can drop the term. Re-organizing the expressions, we then have:

$$\operatorname{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(k)}) > \operatorname{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(k+1)}) + \eta \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x} \rangle$$

Now, notice that if we can ensure that for all $k \in \mathbb{N}_+$, there exists a $\boldsymbol{x}^* \in \mathcal{SVI}(\mathcal{X}, \boldsymbol{f})$ s.t. $\langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^* \rangle \geq 0$, then we have: $\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(k)}) \geq \operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(k+1)})$, implying that $\boldsymbol{x}^{(k)} \to \boldsymbol{x}^*$. converges to a strong solution. Since we cannot assume the existence of a weak solution (i.e., the Minty condition), the next best way to ensure that there $\boldsymbol{x}^* \in \mathcal{SVI}(\mathcal{X}, \boldsymbol{f})$ s.t. $\langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^* \rangle \geq 0$, is to initialize the algorithm with an initial iterate $\boldsymbol{x}^{(0)} \in \mathcal{X}$ which is $O(\delta)$-close to a δ-local weak solution $\boldsymbol{x}^* \in \mathcal{LMVI}^{\delta}(\mathcal{X}, \boldsymbol{f})$[6] for some $\delta \geq 0$, and ensure that all subsequent intermediary iterates $\{\boldsymbol{x}^{(k+0.5)}\}_{k \in \mathbb{N}_{++}}$ remain δ-close to $\boldsymbol{x}^*$.

To ensure this, we have to first bound the distance between the intermediary $\{\boldsymbol{x}^{(k+0.5)}\}_{k \in \mathbb{N}_+}$ and terminal $\{\boldsymbol{x}^{(k)}\}_{k \in \mathbb{N}_+}$ iterates. The following lemma provides us with such a bound.

Lemma 4.3.3 [Distance bound on intermediate iterates]**.**

Let $(\mathcal{X}, \boldsymbol{f})$ be a λ-Lipschitz-continuous VI satisfying the Minty condition, and h a 1-strongly-convex and κ-Lipschitz-smooth kernel function. Consider the mirror extragradient algorithm (Algorithm 3) run with the VI $(\mathcal{X}, \boldsymbol{f})$, the kernel function h, any step size $\eta \geq 0$, for any time horizon $\tau \in \mathbb{N}$, and outputs $\{\boldsymbol{x}^{(t+0.5)}, \boldsymbol{x}^{(t+1)}\}_t$. We then have:

$$\left\| \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k)} \right\| \leq \eta \ell \tag{4.32}$$

where $\ell \doteq \max_{\boldsymbol{x} \in \mathcal{X}} \|\boldsymbol{f}(\boldsymbol{x})\|$.

[6] Note that a local weak solution is guaranteed to be strong solution by Proposition 3.1 of Aussel and Chaipunya (2024).

Proof of Lemma 4.3.3

Note that for all $k \in \mathbb{N}_+$, by the first order optimality conditions of $\boldsymbol{x}^{(k+0.5)}$, we have for all $\boldsymbol{x} \in \mathcal{X}$:

$$\langle \boldsymbol{f}(\boldsymbol{x}^{(k)}) + \frac{1}{\eta}\langle \nabla h(\boldsymbol{x}^{(k+0.5)}) - \nabla h(\boldsymbol{x}^{(k)}), \boldsymbol{x} - \boldsymbol{x}^{(k+0.5)}\rangle \geq 0.$$

Substituting $\boldsymbol{x} = \boldsymbol{x}^{(k)}$ above, we have:

$$\begin{aligned}\langle \boldsymbol{f}(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k+0.5)}\rangle &\geq \frac{1}{\eta}\langle \nabla h(\boldsymbol{x}^{(k)}) - \nabla h(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k+0.5)}\rangle \\ &= \frac{1}{\eta}\left(\mathrm{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) + \mathrm{div}_h(\boldsymbol{x}^{(k)}, \boldsymbol{x}^{(k+0.5)})\right). \end{aligned} \tag{4.33}$$

where the last line was obtained by Lemma 4.3.1.

Re-organizing:

$$\mathrm{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) \leq \eta\langle \boldsymbol{f}(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k+0.5)}\rangle - \mathrm{div}_h(\boldsymbol{x}^{(k)}, \boldsymbol{x}^{(k+0.5)}). \tag{4.34}$$

$$\leq \eta\langle \boldsymbol{f}(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k+0.5)}\rangle - {}^1\!/_2 \left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k+0.5)}\right\|^2 \tag{4.35}$$

$$\leq \eta \left\|\boldsymbol{f}(\boldsymbol{x}^{(k)})\right\| \left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k+0.5)}\right\| - {}^1\!/_2 \left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k+0.5)}\right\|^2 \tag{4.36}$$

$$\leq \eta\ell \left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k+0.5)}\right\| - {}^1\!/_2 \left\|\boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k+0.5)}\right\|^2 \tag{4.37}$$

Since for all $z \in \mathbb{R}$, $ab \in \mathbb{R}_+$, we have $az - bz^2 \leq {}^{a^2}\!/_{4b}$:

$$\mathrm{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) \leq \frac{\eta^2\ell^2}{2} \tag{4.38}$$

Additionally, note that by strong convexity of h, we have $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$, $\mathrm{div}_h(\boldsymbol{x}, \boldsymbol{y}) \geq {}^1\!/_2\|\boldsymbol{x} - \boldsymbol{y}\|^2$ or equivalently $\sqrt{2\mathrm{div}_h(\boldsymbol{x}, \boldsymbol{y})} \geq \|\boldsymbol{x} - \boldsymbol{y}\|^2$. Hence, continuing:

$$\left\|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k)}\right\| \leq \sqrt{2\mathrm{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})} \leq \eta\ell \tag{4.39}$$

With the above Lemma in hand, we can show that if the initial iterate starts close enough to some local weak solution, then the intermediate iterates will remain within this δ-ball for the remainder of the algorithm for an appropriate choice of step size.

Lemma 4.3.4 [Mirror Extragradient Iterates Remain Local]**.**

Let $(\mathcal{X}, \boldsymbol{f})$ be a λ-Lipschitz-continuous VI satisfying the Minty condition, and h a 1-strongly-convex kernel function. Define $\ell \doteq \max_{\boldsymbol{x}\in\mathcal{X}} \|\boldsymbol{f}(\boldsymbol{x})\|$. Suppose that for some $\boldsymbol{x}^* \in \mathcal{LMVI}^\delta(\mathcal{X}, \boldsymbol{f})$ δ-local weak solution, the initial iterate $\boldsymbol{x}^{(0)} \in \mathcal{X}$ is chosen so that $\sqrt{2\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})} \leq \delta - \eta\ell$. Consider the mirror extragradient algorithm (Algorithm 3) run with the VI $(\mathcal{X}, \boldsymbol{f})$, the kernel function h, a step size $\eta \geq 0$, initial iterate $\boldsymbol{x}^{(0)}$, some time horizon $\tau \in \mathbb{N}$, and outputs $\{\boldsymbol{x}^{(t+0.5)}, \boldsymbol{x}^{(t+1)}\}_t$. Then. for all $t \in [\tau]$, we have

$$\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(t)}) \leq 1/2(\delta - \eta\ell)^2 \qquad \text{and } \left\|\boldsymbol{x}^{(t+0.5)} - \boldsymbol{x}^*\right\| \leq \delta \ .$$

Proof of Lemma 4.3.4

We will prove the claim by induction on $t \in \mathbb{N}_+$.

Base case: $t = 0$

$$\begin{aligned}
\left\|\boldsymbol{x}^{(0.5)} - \boldsymbol{x}^*\right\| &= \left\|\boldsymbol{x}^{(0.5)} - \boldsymbol{x}^{(0)} + \boldsymbol{x}^{(0)} - \boldsymbol{x}^*\right\| \\
&\leq \left\|\boldsymbol{x}^{(0.5)} - \boldsymbol{x}^{(0)}\right\| + \left\|\boldsymbol{x}^{(0)} - \boldsymbol{x}^*\right\| \\
&\leq \left\|\boldsymbol{x}^{(0.5)} - \boldsymbol{x}^{(0)}\right\| + \sqrt{2\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})} \\
&\leq \eta\ell + (\delta - \eta\ell) && \text{(Lemma 4.3.3)} \\
&\leq \delta
\end{aligned}$$

Inductive step: Suppose that for all $t = 0, \ldots, \tau$, $\left\|\boldsymbol{x}^{(t+0.5)} - \boldsymbol{x}^*\right\| \leq \delta$ and $\sqrt{2\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(t)})} \leq \delta - \eta\ell$ (or equivalently, $\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(t)}) \leq 1/2(\delta - \eta\ell)^2$). We will show that $\left\|\boldsymbol{x}^{(\tau+1.5)} - \boldsymbol{x}^*\right\| \leq \delta$ and $\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(\tau+1)}) \leq 1/2(\delta - \eta\ell)^2$.

By Lemma 4.3.2, we have:

$$\mathrm{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(\tau)}) - \mathrm{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(\tau+1)}) \geq \eta\langle \boldsymbol{f}(\boldsymbol{x}^{(\tau+0.5)}), \boldsymbol{x}^{(\tau+0.5)} - \boldsymbol{x}\rangle + \left(1 - (\eta\lambda)^2\right) \mathrm{div}_h(\boldsymbol{x}^{(\tau+0.5)}, \boldsymbol{x}^{(\tau)}) \tag{4.40}$$

Substituting in $\boldsymbol{x} \doteq \boldsymbol{x}^* \in \mathcal{LMVI}^\delta(\mathcal{X}, \boldsymbol{f})$, we have:

$$\begin{aligned}
\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(\tau)}) - \mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(\tau+1)}) &\geq \eta \underbrace{\langle \boldsymbol{f}(\boldsymbol{x}^{(\tau+0.5)}), \boldsymbol{x}^{(\tau+0.5)} - \boldsymbol{x}^*\rangle}_{\geq 0} + \left(1 - (\eta\lambda)^2\right) \underbrace{\mathrm{div}_h(\boldsymbol{x}^{(\tau+0.5)}, \boldsymbol{x}^{(\tau)})}_{\geq 0} \\
&\geq 0
\end{aligned}$$

Re-organizing, and using the inductive assumption that $\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(\tau)}) \leq 1/2(\delta - \eta\ell)^2$ we have:

$$1/2(\delta - \eta\ell)^2 \geq \mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(\tau)}) \geq \mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(\tau+1)}) \tag{4.41}$$

Now, notice that we have:

$$\begin{aligned}
\left\|\boldsymbol{x}^{(\tau+0.5)} - \boldsymbol{x}^*\right\| &= \left\|\boldsymbol{x}^{(\tau+0.5)} - \boldsymbol{x}^{(\tau)} + \boldsymbol{x}^{(\tau)} - \boldsymbol{x}^*\right\| \\
&\leq \left\|\boldsymbol{x}^{(\tau+0.5)} - \boldsymbol{x}^{(\tau)}\right\| + \left\|\boldsymbol{x}^{(\tau)} - \boldsymbol{x}^*\right\| \\
&\leq \left\|\boldsymbol{x}^{(\tau+0.5)} - \boldsymbol{x}^{(\tau)}\right\| + \sqrt{2\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(\tau)})} \\
&\leq \eta\ell + (\delta - \eta\ell) && \text{(Lemma 4.3.3)} \\
&\leq \delta
\end{aligned}$$

With the above lemma in hand, modifying the proof of Theorem 4.3.1 slightly, we can show local convergence to a strong solution when the initial iterate of the algorithm is initialized close enough to a local solution.

Theorem 4.3.2 [Mirror Extragradient Method Local Convergence]**.**

Let $(\mathcal{X}, \boldsymbol{f})$ be a λ-Lipschitz-continuous VI, h a 1-strongly-convex and κ-Lipschitz-smooth kernel function, and let $\eta \in \left(0, \frac{1}{\sqrt{2}\lambda}\right]$. Suppose that for some $\boldsymbol{x}^* \in \mathcal{LMVI}^\delta(\mathcal{X}, \boldsymbol{f})$ δ-local weak solution, the initial iterate $\boldsymbol{x}^{(0)} \in \mathcal{X}$ is chosen so that $\sqrt{2\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})} \leq \delta - \eta\ell$.

Consider the mirror extragradient algorithm (Algorithm 3) run with the VI $(\mathcal{X}, \boldsymbol{f})$, the kernel function h, the step size η, initial iterate $\boldsymbol{x}^{(0)}$, an arbitrary time horizon $\tau \in \mathbb{N}$, and outputs $\{\boldsymbol{x}^{(t+0.5)}, \boldsymbol{x}^{(t+1)}\}_t$. Then, we have:

$$\min_{k=0,\dots,\tau} \max_{\boldsymbol{x}\in\mathcal{X}} \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}\rangle \leq \frac{\sqrt{2}(1+\kappa)\mathrm{diam}(\mathcal{X})}{\eta} \frac{\delta}{\sqrt{\tau}}$$

In addition, let $\boldsymbol{x}_{\text{best}}^{(\tau)} \in \arg\min_{\boldsymbol{x}^{(k+0.5)}:k=0,\dots,\tau} \|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k)}\|$. Then, for some choice of $\tau \in O(\frac{1}{\varepsilon^2})$, $\boldsymbol{x}_{\text{best}}^{(\tau)}$ is a ε-strong solution of $(\mathcal{X}, \boldsymbol{f})$.

Proof of Theorem 4.3.2

Taking Lemma 4.3.2 with $\boldsymbol{x} \doteq \boldsymbol{x}^*$, where $\boldsymbol{x}^*$ is given as in the Theorem statement, then by Lemma 4.3.4 we have for all $k \in \mathbb{N}$:

$$\begin{aligned}
\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(k)}) - \mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(k+1)}) &\geq \eta \underbrace{\langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^*\rangle}_{\geq 0} + \left(1 - (\eta\lambda)^2\right) \mathrm{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) \\
&\geq \left(1 - (\eta\lambda)^2\right) \mathrm{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})
\end{aligned}$$

Multiplying both sides by $\left(1-(\eta\lambda)^2\right)^{-1} > 0$, we have:

$$\mathrm{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) \leq \frac{1}{1-(\eta\lambda)^2}\left(\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(k)}) - \mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(k+1)})\right) \tag{4.42}$$

Summing up for $k = 0, \ldots, \tau$:

$$\begin{aligned}\sum_{k=0}^{\tau}\mathrm{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) &\leq \frac{1}{1-(\eta\lambda)^2}\sum_{k=0}^{\tau}\left(\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(k)}) - \mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(k+1)})\right) &(4.43)\\ &\leq \frac{1}{1-(\eta\lambda)^2}\left(\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)}) - \mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(\tau+1)})\right) &(4.44)\\ &\leq \frac{1}{1-(\eta\lambda)^2}\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)}) &(4.45)\end{aligned}$$

Dividing both sides by τ, we have:

$$\frac{1}{\tau}\sum_{k=0}^{\tau}\mathrm{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) \leq \frac{1}{\tau(1-(\eta\lambda)^2)}\left(\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})\right) \tag{4.46}$$

$$\min_{k=0,\ldots,\tau}\mathrm{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)}) \leq \frac{1}{\tau(1-(\eta\lambda)^2)}\left(\mathrm{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})\right) \tag{4.47}$$

We can transform this convergence into a convergence in terms of the primal gap function. Now, recall by the first order optimality conditions of $\boldsymbol{x}^{(k+0.5)}$, we have for all $\boldsymbol{x} \in \mathcal{X}$:

$$\langle \boldsymbol{f}(\boldsymbol{x}^{(k)}) + \frac{1}{\eta}\langle \nabla h(\boldsymbol{x}^{(k+0.5)}) - \nabla h(\boldsymbol{x}^{(k)}), \boldsymbol{x} - \boldsymbol{x}^{(k+0.5)}\rangle \geq 0.$$

Re-organizing, for all $\boldsymbol{x} \in \mathcal{X}$, and $k \in \mathbb{N}$ we have:

$$\begin{aligned}\langle \boldsymbol{f}(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}\rangle &\leq \frac{1}{\eta}\left\|\nabla h(\boldsymbol{x}^{(k+0.5)}) - \nabla h(\boldsymbol{x}^{(k)})\right\|\left\|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}\right\| &(4.48)\\ &\leq \frac{\mathrm{diam}(\mathcal{X})}{\eta}\left\|\nabla h(\boldsymbol{x}^{(k+0.5)}) - \nabla h(\boldsymbol{x}^{(k)})\right\| &(4.49)\\ &\leq \frac{\mathrm{diam}(\mathcal{X})\kappa}{\eta}\left\|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k)}\right\| &(4.50)\end{aligned}$$

where the last line follow from h being κ-Lipschitz-smooth.

Now, with the above inequality in hand, notice that for all $\boldsymbol{x} \in \mathcal{X}$ and $k \in \mathbb{N}$, we have:

$$\begin{aligned}\langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}\rangle &= \langle \boldsymbol{f}(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}\rangle + \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}) - \boldsymbol{f}(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}\rangle\\ &\leq \frac{\mathrm{diam}(\mathcal{X})\kappa}{\eta}\|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k)}\| + \|\boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}) - \boldsymbol{f}(\boldsymbol{x}^{(k)})\| \cdot \|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}\|\\ &\leq \frac{\mathrm{diam}(\mathcal{X})\kappa}{\eta}\|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k)}\| + \lambda\|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k)}\| \cdot \|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}\|\\ &\leq \mathrm{diam}(\mathcal{X})\left(\frac{\kappa}{\eta} + \lambda\right)\|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k)}\|\end{aligned}$$

where the penultimate line follows from the λ-Lipschitz-continuity of $\boldsymbol{f}$, and the strong convexity of h, which means that we have $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$, $\operatorname{div}_h(\boldsymbol{x}, \boldsymbol{y}) \geq 1/2 \|\boldsymbol{x} - \boldsymbol{y}\|^2$..

Now, let $k^* \in \arg\min_{k=0,\dots,\tau} \|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k)}\|$, we then have:

$$
\begin{aligned}
\langle \boldsymbol{f}(\boldsymbol{x}^{(k^*+0.5)}), \boldsymbol{x}^{(k^*+0.5)} - \boldsymbol{x} \rangle &\leq \operatorname{diam}(\mathcal{X}) \left(\frac{\kappa}{\eta} + \lambda \right) \|\boldsymbol{x}^{(k^*+0.5)} - \boldsymbol{x}^{(k^*)}\| \\
&= \operatorname{diam}(\mathcal{X}) \left(\frac{\kappa}{\eta} + \lambda \right) \min_{k=0,\dots,\tau} \|\boldsymbol{x}^{(k^*+0.5)} - \boldsymbol{x}^{(k^*)}\| \\
&\leq \operatorname{diam}(\mathcal{X}) \left(\frac{\kappa}{\eta} + \lambda \right) \min_{k=0,\dots,\tau} \sqrt{2\operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})}
\end{aligned}
$$

where the last line follows from

Now, plugging Equation (4.47) in the above, we have:

$$
\begin{aligned}
\langle \boldsymbol{f}(\boldsymbol{x}^{(k^*+0.5)}), \boldsymbol{x}^{(k^*+0.5)} - \boldsymbol{x} \rangle &\leq \operatorname{diam}(\mathcal{X}) \left(\frac{\kappa}{\eta} + \lambda \right) \min_{k=0,\dots,\tau} \sqrt{2\operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})} \\
&\leq \sqrt{2}\operatorname{diam}(\mathcal{X}) \left(\frac{\kappa}{\eta} + \lambda \right) \sqrt{\min_{k=0,\dots,\tau} \operatorname{div}_h(\boldsymbol{x}^{(k+0.5)}, \boldsymbol{x}^{(k)})} \\
&\leq \frac{\sqrt{2}\operatorname{diam}(\mathcal{X}) \left(\frac{\kappa}{\eta} + \lambda \right)}{\sqrt{1 - (\eta\lambda)^2}} \frac{\sqrt{\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})}}{\sqrt{\tau}}
\end{aligned}
$$

Now, by the assumption that $\eta \leq \frac{1}{\sqrt{2}\lambda} \leq \frac{1}{\lambda}$, we have:

$$
\begin{aligned}
\langle \boldsymbol{f}(\boldsymbol{x}^{(k^*+0.5)}), \boldsymbol{x}^{(k^*+0.5)} - \boldsymbol{x} \rangle &\leq \frac{\sqrt{2}\operatorname{diam}(\mathcal{X}) \left(\frac{\kappa}{\eta} + \frac{1}{\eta} \right)}{\sqrt{1 - (\eta\lambda)^2}} \frac{\sqrt{\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})}}{\sqrt{\tau}} \\
&= \frac{(1+\kappa)\sqrt{2}\operatorname{diam}(\mathcal{X})}{\eta\sqrt{1 - (\eta\lambda)^2}} \frac{\sqrt{\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})}}{\sqrt{\tau}} \\
&= \frac{(1+\kappa)\sqrt{2}\operatorname{diam}(\mathcal{X})}{\eta\sqrt{1 - (1/\sqrt{2})^2}} \frac{\sqrt{\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})}}{\sqrt{\tau}} \\
&= \frac{2(1+\kappa)\operatorname{diam}(\mathcal{X})}{\eta} \frac{\sqrt{\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})}}{\sqrt{\tau}} \\
&= \frac{\sqrt{2}(1+\kappa)\operatorname{diam}(\mathcal{X})}{\eta} \frac{\delta}{\sqrt{\tau}}
\end{aligned}
$$

where the last line follows from the assumption that $\sqrt{2\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})} \leq \delta - \eta\ell$ which implies $\sqrt{\operatorname{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})} \leq \frac{\delta}{\sqrt{2}}$.

That is, we have:

$$
\max_{\boldsymbol{x} \in \mathcal{X}} \langle \boldsymbol{f}(\boldsymbol{x}^{(k+0.5)}), \boldsymbol{x}^{(k+0.5)} - \boldsymbol{x} \rangle \leq \langle \boldsymbol{f}(\boldsymbol{x}^{(k^*+0.5)}), \boldsymbol{x}^{(k^*+0.5)} - \boldsymbol{x} \rangle \leq \frac{\sqrt{2}(1+\kappa)\operatorname{diam}(\mathcal{X})}{\eta} \frac{\delta}{\sqrt{\tau}}
$$

In addition, for any $\varepsilon \geq 0$, letting $\frac{\sqrt{2}(1+\kappa)\mathrm{diam}(\mathcal{X})}{\eta} \frac{\delta}{\sqrt{\tau}} \leq \varepsilon$, and solving for τ, we have:

$$\frac{\sqrt{2}(1+\kappa)\mathrm{diam}(\mathcal{X})}{\eta} \frac{\delta}{\sqrt{\tau}} \leq \varepsilon$$
$$\frac{2(1+\kappa)^2\mathrm{diam}(\mathcal{X})^2}{\eta^2} \frac{\delta^2}{\varepsilon^2} \leq \tau$$

That is, $\boldsymbol{x}^{(\tau)}_{\mathrm{best}} \in \arg\min_{\boldsymbol{x}^{(k+0.5)}:k=0,\dots,\tau} \|\boldsymbol{x}^{(k+0.5)} - \boldsymbol{x}^{(k)}\|$ is a ε-strong solution after $\frac{2(1+\kappa)^2\mathrm{diam}(\mathcal{X})^2}{\eta^2} \frac{\delta}{\varepsilon^2}$ iterations of the mirror extragradient algorithm.

4.4 Merit Function Methods

4.4.1 Merit Function Minimization via Second-Order Methods

Unfortunately, beyond domains where a local Minty solution might not exist, it is not possible to show even local convergence of the extragradient method for VIs. To see this clearly, consider the following example:

Example 4.4.1 [Non-convergence in the absence of local Minty solution]**.**
Consider the VI $(\mathcal{X}, \boldsymbol{f})$ where $\mathcal{X} \doteq \mathbb{R} \times \mathbb{R}$ and $\boldsymbol{f}(x, y) \doteq (x - y, x - y)$. The set of strong solutions of this VI is given by $\mathcal{SVI}(\mathcal{X}, \boldsymbol{f}) = \{(x, y) \in \mathcal{X} \mid x = y\}$. Notice that for any $x^{(0)} > y^{(0)}$, for any choice of step sizes, the iterates generated by the mirror (extra)gradient method tend to infinity, while for $x^{(0)} < y^{(0)}$, the iterates tend to negative infinity.

To overcome this non-convergence issue, given a VI $(\mathcal{X}, \boldsymbol{f})$, we will instead consider second-order methods. Our approach to derive a second order method method will be to optimize a merit function associated with the VI.

Definition 4.4.1 [Merit functions]**.**
Given a VI $(\mathcal{X}, \boldsymbol{f})$. A function $\Xi : \mathcal{X} \to \mathbb{R}$ is said to be a **merit function** for the set of strong (resp. weak) solutions of $(\mathcal{X}, \boldsymbol{f})$ iff

1. for all $\boldsymbol{x} \in \mathcal{X}$, $\Xi(\boldsymbol{x}) \geq 0$
2. $\arg\min_{\boldsymbol{x} \in \mathcal{X}} \Xi(\boldsymbol{x}) = \mathcal{SVI}(\mathcal{X}, \boldsymbol{f})$ (resp. $\arg\min_{\boldsymbol{x} \in \mathcal{X}} \Xi(\boldsymbol{x}) = \mathcal{MVI}(\mathcal{X}, \boldsymbol{f})$)

The canonical merit function associated with the strong solution of any VI is the **primal gap function** $\Xi : \mathcal{X} \to \mathbb{R}_+$:

$$\Xi(\boldsymbol{x}) \doteq \max_{\boldsymbol{y} \in \mathcal{X}} \langle \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x} - \boldsymbol{y} \rangle \tag{4.51}$$

Analogously, we can also define a merit function known as the **dual gap function** $\chi : \mathcal{X} \to \mathbb{R}_+$ associated with weak solutions:

$$\chi(\boldsymbol{x}) \doteq \max_{\boldsymbol{y} \in \mathcal{X}} \langle \boldsymbol{f}(\boldsymbol{y}), \boldsymbol{x} - \boldsymbol{y} \rangle \tag{4.52}$$

An important property of the primal (resp. dual) gap function is that its set of minima coincide with the set of strong (resp. weak) solutions, i.e., $\mathcal{SVI}(\mathcal{X}, \boldsymbol{f}) = \arg\min_{\boldsymbol{x} \in \mathcal{X}} \Xi(\boldsymbol{x})$ (resp. $\mathcal{MVI}(\mathcal{X}, \boldsymbol{f}) = \arg\min_{\boldsymbol{x} \in \mathcal{X}} \chi(\boldsymbol{x})$)

(see, for instance Proposition 2.3 and 2.4 of Huang and Zhang (2023)). In other words, the primal (resp. dual) gap function is a merit function for the strong (resp. weak) solutions of the VI. While the primal gap function is in general non-convex and non-differentiable, the dual gap function is always convex, nevertheless its evaluation requires solving a non-convex optimization problem. As such, finding even stationary points of the primal and dual gap functions is in general intractable.

Nevertheless, it is possible to formulate a differentiable merit function for strong solutions called the α**-regularized primal function** $\Xi_\alpha : \mathcal{X} \to \mathbb{R}_+$:

$$\Xi_\alpha(\boldsymbol{x}) \doteq \max_{\boldsymbol{y} \in \mathcal{X}} \langle \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x} - \boldsymbol{y} \rangle - \frac{\alpha}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2 \tag{4.53}$$

where $\alpha > 0$ is a regularization parameter.

We note the following important lemma due to Fukushima (1992), which we include here for completeness.

Lemma 4.4.1 [Properties of the regularized primal gap]**.**
Consider a continuous VI $(\mathcal{X}, \boldsymbol{f})$. Suppose that $\alpha > 0$, then $\max_{\boldsymbol{y} \in \mathcal{X}} \langle \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x} - \boldsymbol{y} \rangle - \frac{\alpha}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2$ has a unique solution. In addition, the following holds:

1. $\boldsymbol{y}^*(\boldsymbol{x}) = \arg\max_{\boldsymbol{y} \in \mathcal{X}} \langle \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x} - \boldsymbol{y} \rangle - \frac{\alpha}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2 \doteq \Pi_{\mathcal{X}} \left[\boldsymbol{x} - \frac{1}{\alpha} \boldsymbol{f}(\boldsymbol{x})\right]$,

2. $\nabla \Xi_\alpha(\boldsymbol{x}) = \boldsymbol{f}(\boldsymbol{x}) - (\nabla \boldsymbol{f}(\boldsymbol{x}) + \alpha \mathbb{I}) (\boldsymbol{y}^*(\boldsymbol{x}) - \boldsymbol{x})$,

3. $\Xi_\alpha(\boldsymbol{x}) = \max_{\boldsymbol{y} \in \mathcal{X}} \frac{\alpha}{2} \left[\left\|\frac{1}{\alpha} \boldsymbol{f}(\boldsymbol{x})\right\|^2 - \left\|\boldsymbol{y} - \left(\boldsymbol{x} - \frac{1}{\alpha} \boldsymbol{f}(\boldsymbol{x})\right)\right\|^2 \right]$

4. For all $\boldsymbol{x} \in \mathcal{X}$, $\Xi_\alpha(\boldsymbol{x}) \geq 0$ and $\mathcal{SVI}(\mathcal{X}, \boldsymbol{f}) = \arg\min_{\boldsymbol{x} \in \mathcal{X}} \Xi_\alpha(\boldsymbol{x})$

Proof of Lemma 4.4.1

For the first part, first, note that by the first order optimality conditions we have for all $\boldsymbol{x} \in \mathcal{X}$:

$$-\boldsymbol{f}(\boldsymbol{x}) - \alpha (\boldsymbol{y}^* - \boldsymbol{x}) \in \{\boldsymbol{z}' \in \mathcal{X} \mid \langle \boldsymbol{z}', \boldsymbol{z} - \boldsymbol{y}^* \rangle \geq 0 \mid \forall \boldsymbol{z} \in \mathcal{X}\}$$

Re-organizing, we have for all $\boldsymbol{x} \in \mathcal{X}$:

$$\begin{aligned}
\boldsymbol{y}^* &\in \boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}) + \{\boldsymbol{z}' \mid \langle \boldsymbol{z}', \boldsymbol{z} - \boldsymbol{y}^* \rangle \geq 0 \mid \forall \boldsymbol{z}', \boldsymbol{z} \in \mathcal{X}\} \\
&\in \{\boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}) + \boldsymbol{z}' \mid \langle \boldsymbol{z}', \boldsymbol{z} - \boldsymbol{y}^* \rangle \geq 0 \mid \forall \boldsymbol{z}', \boldsymbol{z} \in \mathcal{X}\} \\
&\in \{\boldsymbol{y}^* \in \mathcal{X} \mid \left\langle \boldsymbol{y}^* - (\boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})), \boldsymbol{z} - \boldsymbol{y}^* \right\rangle \geq 0 \mid \forall \boldsymbol{z} \in \mathcal{X}\} \\
&\in \{\boldsymbol{y}^* \in \mathcal{X} \mid \left\langle (\boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})) - \boldsymbol{y}^*, \boldsymbol{z} - \boldsymbol{y}^* \right\rangle \leq 0 \mid \forall \boldsymbol{z} \in \mathcal{X}\} \\
&\in \arg\min_{\boldsymbol{y} \in \mathcal{X}} \left\| \boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{y} \right\|^2
\end{aligned}$$

For the second part of the lemma, applying Danskin's Theorem (Danskin, 1966), we have:

$$\begin{aligned}
\nabla \Xi_\alpha(\boldsymbol{x}) &\doteq (\boldsymbol{f}(\boldsymbol{x}) + \langle \nabla \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x} \rangle) - \langle \nabla \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{y}^*(\boldsymbol{x}) \rangle - \alpha \left(\boldsymbol{y}^*(\boldsymbol{x}) - \boldsymbol{x}\right) \\
&= \boldsymbol{f}(\boldsymbol{x}) - \langle \nabla \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{y}^*(\boldsymbol{x}) - \boldsymbol{x} \rangle - \alpha \left(\boldsymbol{y}^*(\boldsymbol{x}) - \boldsymbol{x}\right) \\
&= \boldsymbol{f}(\boldsymbol{x}) - \left(\nabla \boldsymbol{f}(\boldsymbol{x}) + \alpha \mathbb{I}\right) \left(\boldsymbol{y}^*(\boldsymbol{x}) - \boldsymbol{x}\right)
\end{aligned}$$

For the third part of the lemma, we have:

$$\Xi_\alpha(\boldsymbol{x}) \tag{4.54}$$
$$= \max_{\boldsymbol{y}\in\mathcal{X}} \langle \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x}-\boldsymbol{y}\rangle - \frac{\alpha}{2}\left\|\boldsymbol{y}-\boldsymbol{x}\right\|^2 \tag{4.55}$$
$$= \langle \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x}-\boldsymbol{y}^*(\boldsymbol{x})\rangle - \frac{\alpha}{2}\left\|\boldsymbol{x}-\boldsymbol{y}^*(\boldsymbol{x})\right\|^2 \tag{4.56}$$
$$= \langle \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x}-\boldsymbol{y}^*(\boldsymbol{x})\rangle - \frac{\alpha}{2}\langle \boldsymbol{x}-\boldsymbol{y}^*(\boldsymbol{x}), \boldsymbol{x}-\boldsymbol{y}^*(\boldsymbol{x})\rangle \tag{4.57}$$
$$= \left\langle \boldsymbol{f}(\boldsymbol{x}) - \frac{\alpha}{2}(\boldsymbol{x}-\boldsymbol{y}^*(\boldsymbol{x})), \boldsymbol{x}-\boldsymbol{y}^*(\boldsymbol{x})\right\rangle \tag{4.58}$$
$$= \frac{\alpha}{2}\left\langle \frac{2}{\alpha}\boldsymbol{f}(\boldsymbol{x}) - (\boldsymbol{x}-\boldsymbol{y}^*(\boldsymbol{x})), \boldsymbol{x}-\boldsymbol{y}^*(\boldsymbol{x})\right\rangle \tag{4.59}$$
$$= \frac{\alpha}{2}\left[\left\langle \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x}-\boldsymbol{y}^*(\boldsymbol{x})\right\rangle + \left\langle \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}) - (\boldsymbol{x}-\boldsymbol{y}^*(\boldsymbol{x})), \boldsymbol{x}-\boldsymbol{y}^*(\boldsymbol{x})\right\rangle\right] \tag{4.60}$$
$$= \frac{\alpha}{2}\left[\left\langle \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x}-\boldsymbol{y}^*(\boldsymbol{x})\right\rangle - \left\langle \left(\boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})\right) - \boldsymbol{y}^*(\boldsymbol{x}), \boldsymbol{x}-\boldsymbol{y}^*(\boldsymbol{x})\right\rangle\right] \tag{4.61}$$
$$= \frac{\alpha}{2}\left[\left\langle \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x}-\boldsymbol{y}^*(\boldsymbol{x})\right\rangle - \left\langle (\boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})) - \boldsymbol{y}^*(\boldsymbol{x}), \boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}) + \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{y}^*(\boldsymbol{x})\right\rangle\right] \tag{4.62}$$
$$= \frac{\alpha}{2}\left[\left\langle \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x}-\boldsymbol{y}^*(\boldsymbol{x})\right\rangle - \left\|(\boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})) - \boldsymbol{y}^*(\boldsymbol{x})\right\|^2 - \left\langle (\boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})) - \boldsymbol{y}^*(\boldsymbol{x}), \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})\right\rangle\right] \tag{4.63}$$
$$= \frac{\alpha}{2}\left[\left\langle \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}), \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})\right\rangle - \left\|(\boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})) - \boldsymbol{y}^*(\boldsymbol{x})\right\|^2\right] \tag{4.64}$$
$$= \frac{\alpha}{2}\left[\left\|\frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})\right\|^2 - \left\|(\boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})) - \boldsymbol{y}^*(\boldsymbol{x})\right\|^2\right] \tag{4.65}$$
$$= \max_{\boldsymbol{y}\in\mathcal{X}} \frac{\alpha}{2}\left[\left\|\frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})\right\|^2 - \left\|\boldsymbol{y} - (\boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}))\right\|^2\right] \tag{4.66}$$

For the final part, first note that we have by the third part of the lemma we have for all $\boldsymbol{x} \in \mathcal{X}$:

$$\Xi_\alpha(\boldsymbol{x}) = \max_{\boldsymbol{y}\in\mathcal{X}} \frac{\alpha}{2}\left[\left\|\frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})\right\|^2 - \left\|\boldsymbol{y} - (\boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}))\right\|^2\right] \geq \frac{\alpha}{2}\left[\left\|\frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})\right\|^2 - \left\|\boldsymbol{x} - (\boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}))\right\|^2\right] \geq 0$$

In addition, note that we can write:

$$\Xi_\alpha(\boldsymbol{x}) = \frac{\alpha}{2}\left[\left\|\frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})\right\|^2 - \left\|(\boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})) - \boldsymbol{y}^*(\boldsymbol{x})\right\|^2\right]$$

Now, note that $\left\|\frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})\right\|^2 = \left\|(\boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})) - \boldsymbol{x}\right\|^2$. Hence, we can have $\Xi_\alpha(\boldsymbol{x}) = 0$ iff $\boldsymbol{y}^*(\boldsymbol{x}) = \boldsymbol{x}$, which by the definition of $\boldsymbol{y}^*(\boldsymbol{x})$ implies $\Xi(\boldsymbol{x}) = \max_{\boldsymbol{y}\in\mathcal{X}} \langle \boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x} - \boldsymbol{y}\rangle = 0$. That is, we have $\mathcal{SVI}(\mathcal{X}, \boldsymbol{f}) = \arg\min_{\boldsymbol{x}\in\mathcal{X}} \Xi_\alpha(\boldsymbol{x})$, proving the final part of the lemma.

The first and second part of this lemma show that the gradient of the regularized primal gap function Ξ_α can be evaluated with a constant number of evaluations of the optimality operator of the VI $\boldsymbol{f}$ and its gradient $\nabla\boldsymbol{f}$. Hence, we can minimize the regularized primal gap function, at least locally, using a gradient descent method. Importantly, as this gradient depends on $\nabla\boldsymbol{f}$, the emerging algorithm which we call the mirror potential algorithm is a second order method (Algorithm 5).

Algorithm 5 Mirror Potential Algorithm

Input: $\Xi_\alpha, h, \tau, \eta, \boldsymbol{x}^{(0)}$

Output: $\{\boldsymbol{x}^{(t)}\}_{t\in[\tau]}$

1: **for** $t = 1, \ldots, \tau$ **do**

2: $\quad \boldsymbol{x}^{(t+1)} \leftarrow \arg\min_{\boldsymbol{x}\in\mathcal{X}} \left\{\left\langle \nabla\Xi_\alpha(\boldsymbol{x}^{(t)}), \boldsymbol{x} - \boldsymbol{x}^{(t)}\right\rangle + \frac{1}{2\eta}\mathrm{div}_h(\boldsymbol{x}, \boldsymbol{x}^{(t)})\right\}$

return $\{\boldsymbol{x}^{(t)}\}_{t\in[\tau]}$

With this algorithm in hand, to prove an asymptotic convergence bound for the mirror potential method, we have to ensure that regularized primal gap function Ξ_α is at a minimum a weakly-concave function. It turns out that when the optimality operator $\boldsymbol{f}$ is Lipschitz-continuous *and* Lipschitz-smooth, we can show that the regularized primal gap function is weakly-concave. To prove this we first prove the following technical lemma, which is a slight variant of Lemma 4.2 of Drusvyatskiy and Paquette (2019) for compositions of functions that results in weakly-convex functions.

Lemma 4.4.2 [Composition of weakly-concave and Lipschitz-smooth functions]**.**

Consider a ℓ-Lipschitz-continuous and ρ-weakly-concave function $h : \mathcal{X} \to \mathbb{R}$, and a λ-Lipschitz-continuous and β-Lipschitz-smooth function $\boldsymbol{c} : \mathcal{X} \to \mathcal{X}$. Then, $\boldsymbol{x} \mapsto h(\boldsymbol{c}(\boldsymbol{x}))$ is $\beta\ell + \rho\lambda^2$-weakly-concave.

Proof of Lemma 4.4.2

Let $\varphi(\boldsymbol{x}) \doteq h(\boldsymbol{c}(\boldsymbol{x}))$.

First, note that we have:

$$\begin{aligned}
&\|\nabla h(\boldsymbol{c}(\boldsymbol{x}))[\boldsymbol{c}(\boldsymbol{y}) - \boldsymbol{c}(\boldsymbol{x})] - \nabla h(\boldsymbol{c}(\boldsymbol{x}))[\nabla \boldsymbol{c}(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x})]\|^2 \\
&= \|\nabla h(\boldsymbol{c}(\boldsymbol{x}))[\boldsymbol{c}(\boldsymbol{y}) - \boldsymbol{c}(\boldsymbol{x})) - \nabla \boldsymbol{c}(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x})]\|^2 \\
&\leq \|\nabla h(\boldsymbol{c}(\boldsymbol{x}))\|^2 \|\boldsymbol{c}(\boldsymbol{y}) - \boldsymbol{c}(\boldsymbol{x}) - \nabla \boldsymbol{c}(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x})\|^2 \\
&\leq \ell^2 \|\boldsymbol{c}(\boldsymbol{y}) - \boldsymbol{c}(\boldsymbol{x}) - \nabla \boldsymbol{c}(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x})\|^2 \\
&\leq \frac{\beta \ell^2}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2
\end{aligned}$$

where the penultimate line follows from the Lipschitz-continuity of h, and the final by the β-Lipschitz-smoothness of $\boldsymbol{c}$, which ensures that: $\|\boldsymbol{c}(\boldsymbol{y}) - \boldsymbol{c}(\boldsymbol{x}) - \nabla \boldsymbol{c}(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x})\| \leq \frac{\beta}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2$.

Combining the above with the ρ-weak-concavity of h, which ensures that $h(\boldsymbol{z}) \leq h(\boldsymbol{z}') + \langle \nabla h(\boldsymbol{z}'), \boldsymbol{z} - \boldsymbol{z}' \rangle + \frac{\rho}{2} \|\boldsymbol{z} - \boldsymbol{z}'\|$, we have:

$$\begin{aligned}
\varphi(\boldsymbol{y}) = h(\boldsymbol{c}(\boldsymbol{y})) &\leq h(\boldsymbol{c}(\boldsymbol{x})) + \langle \nabla h(\boldsymbol{c}(\boldsymbol{x})), \boldsymbol{c}(\boldsymbol{y}) - \boldsymbol{c}(\boldsymbol{x}) \rangle + \frac{\rho}{2} \|\boldsymbol{c}(\boldsymbol{y}) - \boldsymbol{c}(\boldsymbol{x})\|^2 \\
&\leq \varphi(\boldsymbol{x}) + \langle \nabla h(\boldsymbol{c}(\boldsymbol{x})), \boldsymbol{c}(\boldsymbol{y}) - \boldsymbol{c}(\boldsymbol{x}) \rangle + \frac{\rho \lambda^2}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2 \\
&\leq \varphi(\boldsymbol{x}) + \langle \nabla h(\boldsymbol{c}(\boldsymbol{x})), \nabla \boldsymbol{c}(\boldsymbol{x})(\boldsymbol{y} - \boldsymbol{x}) \rangle^2 + \frac{\beta \ell}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \frac{\rho \lambda^2}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2 \\
&= \varphi(\boldsymbol{x}) + \langle \nabla \boldsymbol{c}(\boldsymbol{x}) \nabla h(\boldsymbol{c}(\boldsymbol{x})), \boldsymbol{y} - \boldsymbol{x} \rangle^2 + \frac{\beta \ell}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2 + \frac{\rho \lambda^2}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2 \\
&= \varphi(\boldsymbol{x}) + \langle \nabla \varphi(\boldsymbol{x}), \boldsymbol{y} - \boldsymbol{x} \rangle^2 + \frac{\beta \ell + \rho \lambda^2}{2} \|\boldsymbol{y} - \boldsymbol{x}\|^2
\end{aligned}$$

With the above lemma in hand, we can then define a class of VIs for which the regularized primal gap function is weakly-concave. To this end, we define the class of Lipschitz-smooth of VIs.

Definition 4.4.2 [Lipschitz-Smooth VIs]**.**

Given a modulus of smoothness $\beta \geq 0$, a VI $(\mathcal{X}, \boldsymbol{f})$ is β-**Lipschitz-smooth** iff $\boldsymbol{f}$ is β-Lipschitz-smooth.

Next, using Lemma 4.4.2 and the above definition, we prove that the regularized primal gap function is weakly-concave in Lipschitz-continuous and Lipschitz-smooth VIs.

Lemma 4.4.3 [Weak-concavity of regularized primal gap]**.**

Consider a λ-Lipschitz-continuous and β-Lipschitz-smooth VI $(\mathcal{X}, \boldsymbol{f})$. Then, for all $\alpha \geq 0$, the regularized primal gap function Ξ_α associated with $(\mathcal{X}, \boldsymbol{f})$ is $(2\beta\alpha\mathrm{diam}(\mathcal{X})^2 + 1 + 2\lambda)$-weakly-concave.

Proof of Lemma 4.4.3

Let $h(\boldsymbol{z}, \boldsymbol{z}'; \boldsymbol{y}) \doteq \frac{\alpha}{2}\left[\|\boldsymbol{z}\|^2 - \|\boldsymbol{y} - \boldsymbol{z}'\|^2\right]$. Notice that for all $\boldsymbol{y} \in \mathcal{X}$, $h(\boldsymbol{z}, \boldsymbol{z}'; \boldsymbol{y}) + \frac{\alpha}{2}\|\boldsymbol{z}\|^2 = -\|\boldsymbol{y} - \boldsymbol{z}'\|^2$, and hence $(\boldsymbol{z}, \boldsymbol{z}') \mapsto h(\boldsymbol{z}, \boldsymbol{z}'; \boldsymbol{y}) + \frac{\alpha}{2}\|\boldsymbol{z}\|^2$ is concave. That is, for all $\boldsymbol{y} \in \mathcal{X}$, $h(\boldsymbol{z}, \boldsymbol{z}'; \boldsymbol{y})$ is α-weakly-concave. In addition, h is $\alpha\mathrm{diam}(\mathcal{X})$-Lipschitz-continuous, since $\|\nabla_{\boldsymbol{z},\boldsymbol{z}'} h(\boldsymbol{z}, \boldsymbol{z}'; \boldsymbol{y})\| = \alpha\|(\boldsymbol{z}, (\boldsymbol{y} - \boldsymbol{z}'))\| \leq 2\alpha\mathrm{diam}(\mathcal{X})$

Let $\boldsymbol{c}(\boldsymbol{x}) \doteq (\frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}), \boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})$. Now, notice that $\boldsymbol{c}$ is $\frac{1+\lambda}{\alpha}$-Lipschitz-continuous since

$$\|\frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}) - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{y})\| \leq \frac{1}{\alpha}\|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{f}(\boldsymbol{y})\| \leq \frac{\lambda}{\alpha}\|\boldsymbol{x} - \boldsymbol{y}\|$$

and

$$\|\boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{y} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{y})\| \tag{4.67}$$

$$= \|\boldsymbol{x} - \boldsymbol{y} + \frac{1}{\alpha}(\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x}))\| \tag{4.68}$$

$$\leq \|\boldsymbol{x} - \boldsymbol{y}\| + \frac{1}{\alpha}\|\boldsymbol{f}(\boldsymbol{y}) - \boldsymbol{f}(\boldsymbol{x})\| \tag{4.69}$$

$$\leq \|\boldsymbol{x} - \boldsymbol{y}\| + \frac{\lambda}{\alpha}\|\boldsymbol{y} - \boldsymbol{x}\| \tag{4.70}$$

$$\leq \frac{1+\lambda}{\alpha}\|\boldsymbol{x} - \boldsymbol{y}\| \tag{4.71}$$

Meaning that $\boldsymbol{c}$ is $\frac{1+2\lambda}{\alpha}$-Lipschitz-continuous since $\|\boldsymbol{c}(\boldsymbol{x}) - \boldsymbol{c}(\boldsymbol{y})\| \leq \|\frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}) - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{y})\| + \|\boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{y} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{y})\| \leq \frac{1+2\lambda}{\alpha}\|\boldsymbol{x} - \boldsymbol{y}\|$

Additionally, notice that by $\boldsymbol{x} \mapsto \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})$ is $\frac{\beta}{\alpha}$-Lipschitz-smooth since $\|\frac{1}{\alpha}\nabla\boldsymbol{f}(\boldsymbol{x}) - \frac{1}{\alpha}\nabla\boldsymbol{f}(\boldsymbol{y})\| \leq \frac{1}{\alpha}\|\nabla\boldsymbol{f}(\boldsymbol{x}) - \nabla\boldsymbol{f}(\boldsymbol{y})\| \leq \frac{\beta}{\alpha}\|\boldsymbol{x} - \boldsymbol{y}\|$. Similarly, $\boldsymbol{x} \mapsto \boldsymbol{x} - \frac{1}{\alpha}\boldsymbol{f}(\boldsymbol{x})$ is $\frac{\beta}{\alpha}$-Lipschitz-smooth since since $\|1 - \frac{1}{\alpha}\nabla\boldsymbol{f}(\boldsymbol{x}) - 1 + \frac{1}{\alpha}\nabla\boldsymbol{f}(\boldsymbol{y})\| \leq \frac{1}{\alpha}\|\nabla\boldsymbol{f}(\boldsymbol{x}) - \nabla\boldsymbol{f}(\boldsymbol{y})\| \leq \frac{\beta}{\alpha}\|\boldsymbol{x} - \boldsymbol{y}\|$. As a result, $\boldsymbol{c}$ is $\frac{2\beta}{\alpha}$-Lipschitz-smooth since $\|\nabla\boldsymbol{c}(\boldsymbol{x}) - \nabla\boldsymbol{c}(\boldsymbol{y})\| \leq \|\frac{1}{\alpha}\nabla\boldsymbol{f}(\boldsymbol{x}) - \frac{1}{\alpha}\nabla\boldsymbol{f}(\boldsymbol{y})\| + \|1 - \frac{1}{\alpha}\nabla\boldsymbol{f}(\boldsymbol{x}) - 1 + \frac{1}{\alpha}\nabla\boldsymbol{f}(\boldsymbol{y})\| \leq \frac{2\beta}{\alpha}\|\boldsymbol{x} - \boldsymbol{y}\|$.

Hence, by Lemma 4.4.2 $\boldsymbol{x} \mapsto h(\boldsymbol{c}(\boldsymbol{x}))$ is $\frac{2\beta}{\alpha}\alpha^2\mathrm{diam}(\mathcal{X})^2 + \frac{1+2\lambda}{\alpha}\alpha = (2\beta\alpha\mathrm{diam}(\mathcal{X})^2 + 1 + 2\lambda)$-weakly-concave since it is the composition of h which is α-weakly-concave and $\alpha\mathrm{diam}(\mathcal{X})$-Lipschitz-continuous, and $\boldsymbol{c}$ which is a $\frac{1+2\lambda}{\alpha}$-Lipschitz-continuous and $\frac{2\beta}{\alpha}$-Lipschitz-smooth function.

4.4.2 Mirror Potential Algorithm for VIs

With Lemma 4.4.3 in hand, we can then analyze the convergence properties of the mirror potential algorithm. Nevertheless, Ξ_α is in general non-convex, and as it is PPAD complete to compute a minimum of Ξ_α in Lipschitz-continuous and Lipschitz-smooth VIs (recall that this would imply that we have computed a strong solution of such a VI which is a PPAD-complete problem Kapron and Samieefar (2024)), we will instead aim to compute a stationary point of Ξ_α (Definition 4.4.3).

Definition 4.4.3 [Stationary point]**.**
For any approximation parameter $\varepsilon \geq 0$, a ε-**stationary point** $\boldsymbol{x}^* \in \mathcal{X}$ of any optimization problem $\min_{\boldsymbol{x}\in\mathcal{X}} h(\boldsymbol{x})$ where $\mathcal{X} \subseteq \mathcal{U}$ is the constraint set and $h : \mathcal{U} \to \mathbb{R}$ is subdifferentiable objective is defined as the set of ε-strong solutions $\mathcal{SVI}(\mathcal{X}, \mathcal{D}h)$ of the VI $(\mathcal{X}, \mathcal{D}h)$.

A 0-stationary point is simply called a **stationary point**.

With this definition, we can state our Theorem, which shows that for Lipschitz-continuous and Lipschitz-smooth VIs, the regularized primal gap function is weakly-concave and as such a stationary point of it can be computed using standard proof techniques.

Theorem 4.4.1 [Mirror potential method convergence]**.**
Let $(\mathcal{X}, \boldsymbol{f})$ be a λ-Lipschitz-continuous and β-Lipschitz-smooth VI, h a 1-strongly-convex kernel function, $\alpha \geq 0$, $\eta \in \left(0, \frac{1}{2(2\beta\alpha \text{diam}(\mathcal{X})^2+1+2\lambda)}\right]$, and $\boldsymbol{x}^{(0)} \in \mathcal{X}$.

Consider the mirror potential algorithm (Algorithm 5) run with the regularized primal gap Ξ_α associated with $(\mathcal{X}, \boldsymbol{f})$, the kernel function h, an arbitrary time horizon $\tau \in \mathbb{N}$, the step size η, the initial iterate $\boldsymbol{x}^{(0)}$, and which outputs $\{\boldsymbol{x}^{(t)}\}_t$. The following convergence bound to a stationary point of Ξ_α then holds:

$$\min_{k=0,1,\dots,\tau-1} \max_{\boldsymbol{x}\in\mathcal{X}} \langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k)} - \boldsymbol{x} \rangle \leq \frac{2\Xi_\alpha(\boldsymbol{x}^{(0)})}{\tau}$$

In addition, let $\boldsymbol{x}^{(\tau)}_{\text{best}} \in \arg\min_{\boldsymbol{x}^{(k)}:k=0,\dots,\tau-1} \max_{\boldsymbol{x}\in\mathcal{X}} \langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k)} - \boldsymbol{x} \rangle$, then, for some choice of $\tau \in O(\frac{1}{\varepsilon})$, $\boldsymbol{x}^{(\tau)}_{\text{best}}$ is a ε-stationary point of Ξ_α.

Proof

For convenience define $\nu \doteq (2\beta\alpha \text{diam}(\mathcal{X})^2 + 1 + 2\lambda)$. By Lemma 4.4.3, note that Ξ_α is ν-weakly-concave.

Now, for all $k \in \mathbb{N}_+$, by the first order optimality conditions of $\boldsymbol{x}^{(k+1)}$, we have for all $\boldsymbol{x} \in \mathcal{X}$ and $k \in \mathbb{N}$:

$$\langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}) + \frac{1}{\eta}\langle \nabla h(\boldsymbol{x}^{(k+1)}) - \nabla h(\boldsymbol{x}^{(k)}), \boldsymbol{x} - \boldsymbol{x}^{(k+1)} \rangle \geq 0.$$

Substituting $\boldsymbol{x} = \boldsymbol{x}^{(k)}$ above, we have for all $k \in \mathbb{N}$:

$$\begin{aligned}
\langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k+1)} \rangle &\geq \frac{1}{\eta}\langle \nabla h(\boldsymbol{x}^{(k+1)}) - \nabla h(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k+1)} \rangle \\
&= \frac{1}{\eta}\left(\text{div}_h(\boldsymbol{x}^{(k+1)}, \boldsymbol{x}^{(k)}) + \text{div}_h(\boldsymbol{x}^{(k)}, \boldsymbol{x}^{(k+1)}) \right).
\end{aligned}$$

where the last line follows from Lemma 4.3.1.

Re-organizing, we have for all $k \in \mathbb{N}$:

$$\text{div}_h(\boldsymbol{x}^{(k+1)}, \boldsymbol{x}^{(k)}) \leq \eta \langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k)} - \boldsymbol{x}^{(k+1)} \rangle - \underbrace{\text{div}_h(\boldsymbol{x}^{(k)}, \boldsymbol{x}^{(k+1)})}_{\geq 0} \tag{4.72}$$

$$\leq -\eta \langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)} \rangle \tag{4.73}$$

Now, by the ν-weak-concavity of Ξ_α, and the 1-strong-convexity of the kernel function h which implies $\forall \boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$, $\text{div}_h(\boldsymbol{x}, \boldsymbol{y}) \geq 1/2 \|\boldsymbol{x} - \boldsymbol{y}\|^2$, we have for all $k \in \mathbb{N}$:

$$\begin{aligned}
&\Xi_\alpha(\boldsymbol{x}^{(k+1)}) \\
&\leq \Xi_\alpha(\boldsymbol{x}^{(k)}) + \langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)} \rangle + \frac{\nu}{2}\left\|\boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)}\right\|^2 \\
&\leq \Xi_\alpha(\boldsymbol{x}^{(k)}) + \langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)} \rangle + \nu \text{div}_h(\boldsymbol{x}^{(k+1)}, \boldsymbol{x}^{(k)}) \\
&\leq \Xi_\alpha(\boldsymbol{x}^{(k)}) + (1 - \nu\eta)\langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k+1)} - \boldsymbol{x}^{(k)} \rangle && \text{(Equation (4.73))} \\
&\leq \Xi_\alpha(\boldsymbol{x}^{(k)}) + (1 - \nu\eta)\max_{\boldsymbol{x} \in \mathcal{X}} \langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x} - \boldsymbol{x}^{(k)} \rangle
\end{aligned}$$

Unrolling the inequality for $k = 0, 1, \ldots, \tau - 1$, we have:

$$\Xi_\alpha(\boldsymbol{x}^{(\tau)}) \leq \Xi_\alpha(\boldsymbol{x}^{(0)}) + (1 - \nu\eta)\sum_{k=0}^{\tau-1} \max_{\boldsymbol{x} \in \mathcal{X}} \langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x} - \boldsymbol{x}^{(k)} \rangle$$

Re-organizing the inequality, and dropping the expression $\Xi_\alpha(\boldsymbol{x}^{(\tau)}) \geq 0$, we have:

$$(1 - \nu\eta) \sum_{k=0}^{\tau-1} \max_{\boldsymbol{x} \in \mathcal{X}} \langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k)} - \boldsymbol{x} \rangle \leq \Xi_\alpha(\boldsymbol{x}^{(0)})$$

Since $\eta \in (0, \frac{1}{2\nu}]$, we have $(1 - \nu\eta) = \left(1 - \frac{1}{2}\right) \geq \frac{1}{2}$. Hence, we have:

$$\frac{1}{2} \sum_{k=0}^{\tau-1} \max_{\boldsymbol{x} \in \mathcal{X}} \langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k)} - \boldsymbol{x} \rangle \leq \Xi_\alpha(\boldsymbol{x}^{(0)})$$

$$\sum_{k=0}^{\tau-1} \max_{\boldsymbol{x} \in \mathcal{X}} \langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k)} - \boldsymbol{x} \rangle \leq 2\Xi_\alpha(\boldsymbol{x}^{(0)})$$

Multiplying both sides by $\frac{1}{\tau}$, and and applying the generalized means inequality, we have:

$$\frac{1}{\tau} \sum_{k=0}^{\tau-1} \max_{\boldsymbol{x} \in \mathcal{X}} \langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k)} - \boldsymbol{x} \rangle \leq 2\Xi_\alpha(\boldsymbol{x}^{(0)})$$

$$\min_{k=0,1,\dots,\tau-1} \max_{\boldsymbol{x} \in \mathcal{X}} \langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k)} - \boldsymbol{x} \rangle \leq \frac{2\Xi_\alpha(\boldsymbol{x}^{(0)})}{\tau}$$

Finally, we can convert this convergence bound to finite-time convergence result for any $\varepsilon \geq 0$, by setting $\min_{k=0,1,\dots,\tau-1} \max_{\boldsymbol{x} \in \mathcal{X}} \langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k)} - \boldsymbol{x} \rangle \leq \frac{2\Xi_\alpha(\boldsymbol{x}^{(0)})}{\tau} \leq \varepsilon$ and solving for τ which implies:

$$\frac{2\Xi_\alpha(\boldsymbol{x}^{(0)})}{\varepsilon} \leq \tau$$

Hence, for some $\tau \in O(1/\varepsilon)$, we have $\min_{k=0,1,\dots,\tau-1} \max_{\boldsymbol{x} \in \mathcal{X}} \langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k)} - \boldsymbol{x} \rangle \leq \varepsilon$

With this theorem in order, we conclude on a remark on its interpretation before turning to the applications of our results.

Remark 4.4.1 [When are stationary points global solutions]**.**
It is well-known that for monotone VIs, stationary points of the regularized primal gap function correspond to strong solutions of the VI (see, for instance, Theorem 3.3. of Fukushima (1992)). As such, the above result implies a strong solution can be computed in polynomial-time via the mirror potential method in monotone, Lipschitz-continuous, and Lipschitz-smooth VIs.

Chapter 5

Walrasian Economies

5.1 Background

An **Walrasian economy** $(m, \mathcal{Z})$ consists of $m \in \mathbb{N}$ **commodities**[1], with any quantity of any given commodity being exchangeable for a quantity of another. The exchange process is governed by a system of valuation called **prices** modeled as a vector $\boldsymbol{p} \in \mathbb{R}^m_+$ s.t. $p_j \geq 0$ is the price of commodity $j \in [m]$.[2] Prices $\boldsymbol{p} \in \mathbb{R}^m_+$ allow the **sale** of $x \in \mathbb{R}_+$ units of any commodity $j \in [m]$ in exchange for the **purchase** of $\frac{xp_j}{p_k}$ units of any other commodity $k \in [m]$.

For any price system, the economy determines quantities of each commodity which can be bought and sold, with all admissible exchanges being summarized by an **excess demand correspondence** $\mathcal{Z} : \mathbb{R}^m_+ \rightrightarrows \mathbb{R}^m$ which for any prices $\boldsymbol{p} \in \mathbb{R}^m_+$ outputs a **set of excess demands** $\mathcal{Z}(\boldsymbol{p}) \subseteq \mathbb{R}^m$ with each excess demand $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p})$. For any price $\boldsymbol{p} \in \mathbb{R}^m_+$ and excess demands $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p})$, $z_j(\boldsymbol{p}) \geq 0$ denotes the number of units of commodity $j \in [m]$ **demanded in excess** (i.e., more units of it are bought than sold) $j \in [m]$, while $z_j(\boldsymbol{p}) < 0$ denotes the number of units of commodity $j \in [m]$ **supplied in excess** (i.e., more units of it are

[1] The "commodity" terminology is used here in the tradition of Arrow and Debreu (1954), and refers to any raw, intermediate, & final commodities, labor & services.

[2] The astute reader might notice that in real-world economies the prices of certain commodities can be negative (e.g., prices of oil when storage of excess oil is not possible), and might rise the concern that the model does not account for the possibility negative prices. However, in these cases the price of the commodity is "negative" only colloquially speaking, and rather the price of an associated commodity is positive. For instance, when the price of oil is negative, companies are no more selling oil, rather they are buying a service: the storage of oil. As such, we include "negative pricing" in the real-world by adding the commodities with "negative prices" as additional commodities into the economy (e.g., including both oil and the sale of oil ascommodities). We will see in Chapter 13, Part III, an example of a Walrasian economy in which commodities are explicitly modeled and which can capture this negative pricing phenomenon.

sold than bought). If $\mathcal{Z}$ is singleton-valued, then we will for convenience represent $\mathcal{Z}$ as a function and denote it $\boldsymbol{z}$.

A price vector $\boldsymbol{p} \in \mathbb{R}_+^m$ is said to be **feasible** if there exists a $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p})$ s.t. for all commodity $j \in [m]$, $z_j(\boldsymbol{p}) \leq 0$. Similarly, a price vector $\boldsymbol{p} \in \mathbb{R}_+^m$ is said to satisfy **Walras' law** if there exists a $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p})$ s.t. $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) = 0$.

The canonical solution concept for Walrasian equilibria is the Walrasian equilibrium (Walras, 1896). In the sequel, we will introduce algorithms with polynomial-time convergence guarantees to a Walrasian equilibrium, and to analyze their convergence will use a computationally relevant generalization of Walrasian equilibrium, namely the approximate Walrasian equilibrium to account for the bounded accuracy of computational methods.

Definition 5.1.1 [Approximate Walrasian Equilibrium]**.**
Given an approximation parameter $\varepsilon \geq 0$, a price vector $\boldsymbol{p}^* \in [0,1]^m$ is said to be a ε-**Walrasian** (or ε-**competitive**) **equilibrium** of a Walrasian economy $(m, \mathcal{Z})$ if there exists an excess demand $\boldsymbol{z}(\boldsymbol{p}^*) \in \mathcal{Z}(\boldsymbol{p}^*)$ s.t.

$$\begin{aligned} &(\varepsilon\text{−Feasibility}) && \text{For all commodities } j \in [m],\ z_j(\boldsymbol{p}^*) \leq \varepsilon \\ &(\varepsilon\text{-Walras' law}) && -\varepsilon \leq \boldsymbol{p}^* \cdot \boldsymbol{z}(\boldsymbol{p}^*) \leq \varepsilon \end{aligned}$$

We denote the set of ε-Walrasian equilibria of any Walrasian economy $(m, \mathcal{Z})$ by $\mathcal{WE}_\varepsilon(m, \mathcal{Z})$.

A 0-Walrasian equilibrium is simply called a **Walrasian equilibrium**, in which case we denote the set of Walrasian equilibria $\mathcal{WE}(m, \mathcal{Z})$.

Seen otherwise, a Walrasian equilibrium $\boldsymbol{p}^* \in \mathbb{R}_+^m$ is a price vector s.t. for all commodities $j \in [m]$, $p_j^* > 0 \implies z_j(\boldsymbol{p}^*) = 0$ and $p_j^* = 0 \implies z_j(\boldsymbol{p}^*) \leq 0$. Intuitively, a Walrasian equilibrium is a price vector which ensures that the exchange of any commodity with another can be implemented. More precisely, on the one hand, if the price of a commodity $j \in [m]$ is strictly positive then at a Walrasian equilibrium commodity j will always find a buyer since its excess demand is zero, which makes sense since the exchange system dictates that j can be exchanged for strictly positive units of some other commodity $k \in [m]$. On the other hand, if the price of commodity j is zero, at a Walrasian equilibrium the commodity might not find a buyer, which makes sense since the price system dictates that the commodity j cannot be exchanged for any other good.

5.2 Walrasian Economies and Variational Inequalities

With definitions in order, we now present the fundamental relationship there exists between Walrasian economies and VIs. We will use this relation to first establish the existence of a Walrasian equilibrium in continuous balanced economies, and then introduce efficient algorithms for the computation of a Walrasian equilibrium in Lipschitz-continuous balanced economies.

5.2.1 Walrasian Economies and Complementarity Problems

The following theorem due to Dafermos (1990), is to the best of our knowledge the first result exposing the connection between VIs and Walrasian equilibria (see, Nagurney (2009) for additional references), and states that the problem of computing a Walrasian equilibrium is equivalent to the problem of computing a strong solution of a VI whose set of constraints is given by the positive ortanth (a class of VIs known as **complementarity problems** Cottle and Dantzig (1968)).

Theorem 5.2.1 [Walrasian economies as Complementarity Problems]**.**
The set of Walrasian equilibria of any Walrasian economy $(m, \mathcal{Z})$ is equal to the set of strong solutions of the VI $(\mathbb{R}^m_+, -\mathcal{Z})$, i.e., $\mathcal{WE}(m, \mathcal{Z}) = \mathcal{SVI}(\mathbb{R}^m_+, -\mathcal{Z})$.

Proof of Theorem 5.2.1

$(\implies)$ Let $\boldsymbol{p}^* \in \mathcal{WE}(m, \mathcal{Z})$ be a Walrasian equilibrium. Then, for some $\boldsymbol{z}(\boldsymbol{p}^*) \in \mathcal{Z}(\boldsymbol{p})$, we have:

$$\begin{aligned}
&\langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} - \boldsymbol{p}^* \rangle && \forall \boldsymbol{p} \in \mathbb{R}^m_+ \\
&= \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} \rangle - \underbrace{\langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* \rangle}_{=0} && \forall \boldsymbol{p} \in \mathbb{R}^m_+ \\
&= \underbrace{\langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} \rangle}_{\leq 0} && \forall \boldsymbol{p} \in \mathbb{R}^m_+ \\
&\leq 0
\end{aligned}$$

where the last line follows from the feasibility of $\boldsymbol{z}(\boldsymbol{p}^*)$, i.e., $\boldsymbol{z}(\boldsymbol{p}^*) \leq 0$, and the positivity of $\boldsymbol{p}$.

($\Longleftarrow$) Let $\boldsymbol{p}^* \in \mathcal{SVI}(\mathbb{R}^m_+, -\mathcal{Z})$. Then, for some $\boldsymbol{z}(\boldsymbol{p}^*) \in \mathcal{Z}(\boldsymbol{p})$, we have:

$$0 \geq \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} - \boldsymbol{p}^* \rangle \qquad \forall \boldsymbol{p} \in \mathbb{R}^m_+$$

Substituting $\boldsymbol{p} \doteq \boldsymbol{p}^* + \boldsymbol{j}_j$, we have:

$$\begin{aligned} 0 &\geq \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* + \boldsymbol{j}_j - \boldsymbol{p}^* \rangle \\ &= \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{j}_j \rangle \\ &\geq z_j(\boldsymbol{p}^*) && \forall j \in [m] \end{aligned}$$

That is, $\boldsymbol{p}^*$ is feasible.

Similarly, substituting in $\boldsymbol{p} \doteq \boldsymbol{0}$ and $\boldsymbol{p} \doteq 2\boldsymbol{p}^*$, we have:

$$0 \leq \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* \rangle$$

and

$$0 \geq \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* \rangle \tag{5.1}$$

That is, $\boldsymbol{p}^*$ satisfies weak Walras' law. Hence, $\boldsymbol{p}^*$ is a Walrasian equilibrium.

5.2.2 Balanced Economies and Variational Inequalities

While Theorem 5.2.1 is useful to approach any Walrasian equilibrium computation problem as a strong solution computation problem, as the domain of prices is unbounded, i.e., $\mathbb{R}^m_+$, to obtain existence and computational results, we have to restrict the class of Walrasian economies we study. To this end, we introduce two important classes of Walrasian economies. The first of these classes are balanced economies.

Definition 5.2.1 [Balanced economies]**.**

A **balanced economy** is a Walrasian economy $(m, \mathcal{Z})$ whose excess demand correspondence satisfies:

(Homogeneity of degree 0) For all $\lambda > 0$, $\mathcal{Z}(\lambda \boldsymbol{p}) = \mathcal{Z}(\boldsymbol{p})$

(Weak Walras' law) For all $\boldsymbol{p} \in \mathbb{R}^m_+$ and $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p})$, $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) \leq 0$

Intuitively, homogeneity requires that prices have a meaning only relative to other prices, and have no absolute meaning of their own (i.e., if all prices get scaled by the same amount the excess demand remains unchanged); weak Walras' law requires budget-balance to hold (i.e., at any given prices, the total value of what is being demanded cannot exceed the value of what is supplied). While homogeneity of degree 0 is a standard assumption, weak Walras' law is significantly weaker than standard assumptions previously considered in the literature (see, for instance Arrow and Hurwicz (1958) and Debreu (1974)), and are satisfied by Arrow-Debreu economies (Arrow and Debreu, 1954) (see, Chapter 10 for additional details).

We now provide a novel characterization of Walrasian equilibrium prices in balanced economies as a VI over $[0,1]^m$ rather than $\mathbb{R}_+^m$ which will allow us to obtain polynomial-time algorithms for the computation of Walrasian equilibrium, as the computational guarantees of our algorithms for VIs depend on the diameter of the constraint space of the VIs. In particular, we will now show that the set of Walrasian equilibria of any balanced economy can be restated as the set of strong solutions of a modified VI $([0,1]^m, -\mathcal{Z})$ where the constraint space is $[0,1]^m$ i.e.:

$$\text{Find } \boldsymbol{p}^* \in [0,1]^m \text{ such that} \qquad \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} - \boldsymbol{p}^* \rangle \leq 0 \qquad \text{for all } \boldsymbol{p} \in [0,1]^m \tag{5.2}$$

$$\text{and for some} \qquad \boldsymbol{z}(\boldsymbol{p}^*) \in \mathcal{Z}(\boldsymbol{p}^*) \tag{5.3}$$

Theorem 5.2.2 [Balanced economies as VIs]**.**

For any balanced economy $(m, \mathcal{Z})$, the set of Walrasian equilibria is equal to the strictly positive cone generated by the strong solutions of the continuous VI $([0,1]^m, -\mathcal{Z})$, i.e., $\mathcal{WE}(m, \mathcal{Z}) = \bigcup_{\lambda \geq 1} \lambda\, \mathcal{SVI}([0,1]^m, -\mathcal{Z})$.

Proof of Theorem 5.2.2

($\implies$) Let $\boldsymbol{p}^* \in \mathcal{WE}(m, \mathcal{Z})$ be a Walrasian equilibrium. Let $\alpha \doteq \frac{1}{\max\{1, \|\boldsymbol{p}^*\|_\infty\}}$. Then, we have $\alpha \boldsymbol{p}^* \in [0,1]^m$. Further, for some $\boldsymbol{z}(\alpha \boldsymbol{p}^*) \in \mathcal{Z}(\alpha \boldsymbol{p}^*)$, we have:

$$\begin{aligned}
&\langle -\boldsymbol{z}(\alpha \boldsymbol{p}^*), \alpha \boldsymbol{p}^* - \boldsymbol{p} \rangle && \forall \boldsymbol{p} \in [0,1]^m \\
&= \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} - \alpha \boldsymbol{p}^* \rangle && \forall \boldsymbol{p} \in [0,1]^m && \text{(Homogeneity of } \boldsymbol{z}) \\
&= \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} \rangle - \alpha \underbrace{\langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* \rangle}_{=0} && \forall \boldsymbol{p} \in [0,1]^m \\
&= \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} \rangle && \forall \boldsymbol{p} \in [0,1]^m \\
&\leq 0
\end{aligned}$$

where the penultimate line follows from Walras' law holding at a Walrasian equilibrium, and the last line follows from the feasibility of $\boldsymbol{z}(\boldsymbol{p}^*)$, i.e., $\boldsymbol{z}(\boldsymbol{p}^*) \leq \boldsymbol{0}$, and the positivity of $\boldsymbol{p}$. Hence, $\alpha \boldsymbol{p}^*$ is a strong solution of the VI $([0,1]^m, -\mathcal{Z})$, which means that $\boldsymbol{p}^* \in \frac{1}{\alpha} \mathcal{SVI}([0,1]^m, -\mathcal{Z})$.

Now, notice that by homogeneity of the excess demand in balanced economies since for all $\lambda > 0$, $\mathcal{Z}(\lambda \boldsymbol{p}^*) = \mathcal{Z}(\boldsymbol{p}^*)$, if $\boldsymbol{p}^*$ is a Walrasian equilibrium, then so is $\lambda \boldsymbol{p}^*$. Hence, α takes values in $(0,1]$, implying $\frac{1}{\alpha} \in [1, \infty)$, and as such we must have $\mathcal{WE}(m, \mathcal{Z}) \subseteq \bigcup_{\lambda \geq 1} \lambda \, \mathcal{SVI}([0,1]^m, -\mathcal{Z})$.

($\impliedby$) Let $\boldsymbol{p}^* \in \mathcal{SVI}([0,1]^m, -\mathcal{Z})$ and $\lambda \geq 1$. Then, for some $\boldsymbol{z}(\boldsymbol{p}^*) \in \mathcal{Z}(\boldsymbol{p}^*)$, we have:

$$\begin{aligned}
0 &\geq \langle -\boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* - \boldsymbol{p} \rangle && \forall \boldsymbol{p} \in [0,1]^m \\
&= \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} - \boldsymbol{p}^* \rangle && \forall \boldsymbol{p} \in [0,1]^m \\
&= \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} \rangle - \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* \rangle && \forall \boldsymbol{p} \in [0,1]^m
\end{aligned} \tag{5.4}$$

Plugging $\boldsymbol{p} = \boldsymbol{0}_m$ in Equation (5.4), we then have:

$$\begin{aligned}
&0 \geq \underbrace{\langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{0}_m \rangle}_{=0} - \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* \rangle \\
&0 \geq -\langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* \rangle \\
&0 \leq \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* \rangle \\
&0 \leq \langle \boldsymbol{z}(\lambda \boldsymbol{p}^*), \boldsymbol{p}^* \rangle && \text{(Homogeneity of } \boldsymbol{z}) \\
&0 \leq \langle \boldsymbol{z}(\lambda \boldsymbol{p}^*), \lambda \boldsymbol{p}^* \rangle
\end{aligned}$$

Further, since $(m, \mathcal{Z})$ is balanced, we have $\lambda \boldsymbol{p}^* \cdot \boldsymbol{z}(\lambda \boldsymbol{p}^*) = \boldsymbol{p}^* \cdot \boldsymbol{z}(\boldsymbol{p}^*) \leq 0$, hence, combining it with the above inequality, we must have $\lambda \boldsymbol{p}^* \cdot \boldsymbol{z}(\lambda \boldsymbol{p}^*) = 0$, meaning that $\lambda \boldsymbol{p}^*$ satisfies Walras' law.

In addition, continuing from Equation (5.4) again, we have:

$$
\begin{aligned}
0 &\geq \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} \rangle - \underbrace{\langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* \rangle}_{\leq 0} \\
&\geq \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} \rangle && \forall \boldsymbol{p} \in [0,1]^m && (5.5) \\
&\geq \langle \boldsymbol{z}(\lambda \boldsymbol{p}^*), \boldsymbol{p} \rangle && \forall \boldsymbol{p} \in [0,1]^m && (5.6)
\end{aligned}
$$

where the penultimate line follows from the fact that balanced economies satisfy weak Walras' law, and the last line from homogeneity of degree 0 of the excess demand.

Now, plugging $\boldsymbol{p} = \boldsymbol{j}_j$ for all $j \in [m]$ in Equation (5.6), we have:

$$
\begin{aligned}
0 &\geq \langle \boldsymbol{z}(\lambda \boldsymbol{p}^*), \boldsymbol{j}_j \rangle && \forall j \in [m] \\
&\geq z_j(\lambda \boldsymbol{p}^*) && \forall j \in [m] \ .
\end{aligned}
$$

That is, $\lambda \boldsymbol{p}^*$ is feasible. Putting it all together, $\lambda \boldsymbol{p}^*$ must be a Walrasian equilibrium. As such we must have $\bigcup_{\lambda \geq 1} \lambda \, \mathcal{SVI}([0,1]^m, -\mathcal{Z}) \subseteq \mathcal{WE}(m, \mathcal{Z})$

In the sequel, we will make use of the following lemma which states that for any balanced economy $(m, \mathcal{Z})$, any approximate strong solution of the VI $([0,1]^m, -\mathcal{Z})$ is an approximate Walrasian equilibrium of $(m, \mathcal{Z})$.

Lemma 5.2.1 [ε-strong solution and ε-Walrasian equilibrium]**.**

For any balanced economy $(m, \mathcal{Z})$, any ε-strong solution of the VI $([0,1]^m, -\mathcal{Z})$ is a ε-Walrasian equilibrium of $(m, \mathcal{Z})$.

Proof of Lemma 5.2.1

(ε-strong solution $\implies$ ε-Walrasian equilibrium) For any $\varepsilon \geq 0$, let $\boldsymbol{p}^* \in \mathcal{SVI}_\varepsilon([0,1]^m, -\mathcal{Z})$. Then, for some $\boldsymbol{z}(\boldsymbol{p}^*) \in \mathcal{Z}(\boldsymbol{p}^*)$, we have:

$$\begin{aligned}
\varepsilon &\geq \langle -\boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* - \boldsymbol{p} \rangle && \forall \boldsymbol{p} \in [0,1]^m \\
&= \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} - \boldsymbol{p}^* \rangle && \forall \boldsymbol{p} \in [0,1]^m \\
&= \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} \rangle - \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* \rangle && \forall \boldsymbol{p} \in [0,1]^m \qquad (5.7)
\end{aligned}$$

Plugging $\boldsymbol{p} = \boldsymbol{0}_m$ in Equation (5.4), we then have:

$$\begin{aligned}
\varepsilon &\geq \underbrace{\langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{0}_m \rangle}_{=0} - \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* \rangle \\
\varepsilon &\geq - \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* \rangle \\
-\varepsilon &\leq \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* \rangle
\end{aligned}$$

Further, since $(m, \mathcal{Z})$ is balanced, we have $\boldsymbol{p}^* \cdot \boldsymbol{z}(\boldsymbol{p}^*) \leq 0 \leq \varepsilon$, hence, combining it with the above inequality, we must have that $\boldsymbol{p}^*$ satisfies ε-Walras' law.

In addition, continuing from Equation (5.7) again, we have:

$$\begin{aligned}
\varepsilon &\geq \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} \rangle - \underbrace{\langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* \rangle}_{\leq 0} \\
&\geq \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} \rangle && \forall \boldsymbol{p} \in [0,1]^m \qquad (5.8)
\end{aligned}$$

where the last line follows from the fact that balanced economies satisfy weak Walras' law.

Now, plugging $\boldsymbol{p} = \boldsymbol{j}_j$ for all $j \in [m]$ in Equation (5.8), we have:

$$\begin{aligned}
\varepsilon &\geq \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{j}_j \rangle && \forall j \in [m] \\
&\geq z_j(\boldsymbol{p}^*) && \forall j \in [m] \ .
\end{aligned}$$

That is, $\boldsymbol{p}^*$ is ε-feasible. Putting it all together, $\boldsymbol{p}^*$ must be a ε-Walrasian equilibrium.

5.2.3 Competitive Economies and Continuous Variational Inequalities

We now turn our attention to prove the existence of Walrasian equilibrium. In balanced economies, under the assumption that the excess demand correspondence $\mathcal{Z}$ is upper hemicontinuous, non-empty-, compact-, and convex-valued, it is possible to prove the existence of a Walrasian equilibrium $\boldsymbol{p}^* \in [0,1]^m$ as a corollary of the existence of strong solutions in continuous VIs (Theorem 4.1.1). Unfortunately, this Walrasian equilibrium can be trivial, i.e., $\boldsymbol{p}^* = \mathbf{0}_m$, and to prove the existence of a non-trivial Walrasian equilibrium, we have to restrict our attention to a canonical subset of balanced Walrasian economies studied in the literature which we call competitive economies (Debreu, 1974; Sonnenschein, 1972).

Definition 5.2.2 [Competitive economy]**.**
A **competitive economy** is a Walrasian economy $(m, \mathcal{Z})$ whose excess demand correspondence satisfies:

(Homogeneity of degree 0)	For all $\lambda > 0$, $\mathcal{Z}(\lambda \boldsymbol{p}) = \mathcal{Z}(\boldsymbol{p})$
(Weak Walras' law)	For all $\boldsymbol{p} \in \mathbb{R}^m_+$ and $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p})$, $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) \leq 0$
(Non-Satiation)	For all $\boldsymbol{p} \in \mathbb{R}^m_+$, and $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p})$, $\boldsymbol{z}(\boldsymbol{p}) \leq \mathbf{0}_m \implies \boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) = 0$

That is, a competitive economy is a balanced economy for which when the excess demand is feasible, then Walras' law holds. Intuitively the additional non-satiation condition requires that whenever all goods are supplied in excess, it must be that the economy cannot spend any more money on purchasing commodities (i.e., the value of the excess demand is 0). As such, the excess demand is non-satiated, in the sense that the economy cannot demand more of any commodity because it cannot afford it, and not because it is not supplied in sufficient quantity. In competitive economies, an alternative VI characterization of Walrasian equilibrium holds over the constraint space Δ_m rather than $[0,1]^m$, which is more suitable for proving existence.

Theorem 5.2.3 [Competitive economies as VIs]**.**
For any competitive economy $(m, \mathcal{Z})$, the set of Walrasian equilibria is equal to the strictly positive cone generated by the strong solutions of the continuous VI $(\Delta_m, -\mathcal{Z})$, i.e., $\mathcal{WE}(m, \mathcal{Z}) = \bigcup_{\lambda \geq 1} \lambda \, \mathcal{SVI}(\Delta_m, -\mathcal{Z})$.

Proof of Theorem 5.2.3

($\implies$) Let $\boldsymbol{p}^* \in \mathcal{WE}(m, \mathcal{Z})$ be a Walrasian equilibrium. Let $\alpha \doteq \frac{1}{\|\boldsymbol{p}^*\|_1}$. Then, we have $\alpha \boldsymbol{p}^* \in \Delta_m$. Further, for some $\boldsymbol{z}(\alpha \boldsymbol{p}^*) \in \mathcal{Z}(\alpha \boldsymbol{p}^*)$, we have:

$$
\begin{aligned}
&\langle -\boldsymbol{z}(\alpha \boldsymbol{p}^*), \alpha \boldsymbol{p}^* - \boldsymbol{p} \rangle && \forall \boldsymbol{p} \in \Delta_m \\
&= \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} - \alpha \boldsymbol{p}^* \rangle && \forall \boldsymbol{p} \in \Delta_m && \text{(Homogeneity of } \boldsymbol{z}) \\
&= \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} \rangle - \alpha \underbrace{\langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* \rangle}_{=0} && \forall \boldsymbol{p} \in \Delta_m \\
&= \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} \rangle && \forall \boldsymbol{p} \in \Delta_m \\
&\leq 0
\end{aligned}
$$

where the penultimate line follows from Walras' law holding at a Walrasian equilibrium, and the last line follows from the feasibility of $\boldsymbol{z}(\boldsymbol{p}^*)$, i.e., $\boldsymbol{z}(\boldsymbol{p}^*) \leq \boldsymbol{0}$, and the positivity of $\boldsymbol{p}$. Hence, $\alpha \boldsymbol{p}^*$ is a strong solution of the VI $(\Delta_m, -\mathcal{Z})$, which means that $\boldsymbol{p}^* \in \frac{1}{\alpha} \mathcal{SVI}(\Delta_m, -\mathcal{Z})$.

Now, notice that by homogeneity of the excess demand in competitive economies since for all $\lambda > 0$, $\mathcal{Z}(\lambda \boldsymbol{p}^*) = \mathcal{Z}(\boldsymbol{p}^*)$, if $\boldsymbol{p}^*$ is a Walrasian equilibrium, then so is $\lambda \boldsymbol{p}^*$. Hence, α takes values in $(0, 1]$, implying $\frac{1}{\alpha} \in [1, \infty)$, and as such we must have $\mathcal{WE}(m, \mathcal{Z}) \subseteq \bigcup_{\lambda \geq 1} \lambda \mathcal{SVI}(\Delta_m, -\mathcal{Z})$.

($\impliedby$) Let $\boldsymbol{p}^* \in \mathcal{SVI}(\Delta_m, -\mathcal{Z})$ and $\lambda \geq 1$. Then, for some $\boldsymbol{z}(\boldsymbol{p}^*) \in \mathcal{Z}(\boldsymbol{p}^*)$, we have:

$$
\begin{aligned}
0 &\geq \langle -\boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* - \boldsymbol{p} \rangle && \forall \boldsymbol{p} \in \Delta_m \\
&= \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} - \boldsymbol{p}^* \rangle && \forall \boldsymbol{p} \in \Delta_m \\
&= \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} \rangle - \underbrace{\langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p}^* \rangle}_{\leq 0} \\
&\geq \langle \boldsymbol{z}(\boldsymbol{p}^*), \boldsymbol{p} \rangle && \forall \boldsymbol{p} \in \Delta_m \\
&\geq \langle \boldsymbol{z}(\lambda \boldsymbol{p}^*), \boldsymbol{p} \rangle && \forall \boldsymbol{p} \in \Delta_m
\end{aligned}
$$

where the penultimate line follows from the fact that competitive economies satisfy weak Walras' law, and the last line from homogeneity of degree 0 of the excess demand.

Now, plugging $\boldsymbol{p} = \boldsymbol{j}_j$ for all $j \in [m]$ in the above, we have:

$$\begin{aligned} 0 &\geq \langle \boldsymbol{z}(\lambda \boldsymbol{p}^*), \boldsymbol{j}_j \rangle && \forall j \in [m] \\ &\geq z_j(\lambda \boldsymbol{p}^*) && \forall j \in [m] \ . \end{aligned}$$

That is, $\lambda \boldsymbol{p}^*$ is feasible. Now by non-satiation, since $\boldsymbol{z}(\lambda \boldsymbol{p}^*) \leq \boldsymbol{0}_m$, we must have $\lambda \boldsymbol{p}^* \cdot \boldsymbol{z}(\lambda \boldsymbol{p}^*) \geq 0$. As by weak Walras' law $\lambda \boldsymbol{p}^* \cdot \boldsymbol{z}(\lambda \boldsymbol{p}^*) \leq 0$, we must have $\lambda \boldsymbol{p}^* \cdot \boldsymbol{z}(\lambda \boldsymbol{p}^*) = 0$, meaning that $\lambda \boldsymbol{p}^*$ satisfies Walras' law. Putting it all together, $\lambda \boldsymbol{p}^*$ must be a Walrasian equilibrium. As such we must have $\bigcup_{\lambda \geq 1} \lambda \, \mathcal{SVI}(\Delta_m, -\mathcal{Z}) \subseteq \mathcal{WE}(m, \mathcal{Z})$

To prove existence, it will be necessary to make assumptions on the continuity of the excess demand, which necessitates the definition of continuous economies. We note that in the following definition we assume upper hemicontinuity only on Δ_m, since in competitive, and more generally balanced, economies it is too restrictive to assume that the excess demand $\mathcal{Z}$ is upper hemicontinuous on $\mathbb{R}^m_+$ since any correspondence which is homogeneous of degree 0 and continuous in the entirety of its domain is constant. Intuitively, continuous economies are those economies in which changes in the proportions of prices lead to well-behaved changes in excess demands.

Definition 5.2.3 [Continuous economies]**.**

A **continuous economy** is a Walrasian economy $(m, \mathcal{Z})$ whose excess demand correspondence $\mathcal{Z}$ is upper hemicontinuous on Δ_m, non-empty-, compact-, and convex-valued.

Remark 5.2.1 [Continuity of excess demand]**.**

In more stylized applications (see, for instance, Chapter 6 or Chapter 10), the excess demand correspondence is in general defined so as to be guaranteed to be continuous only on the interior of the unit simplex, i.e., $\text{int}(\Delta_m) = \Delta_m$, as the excess demand for a good can be infinite if the price of any goods is 0. However, this issue in these stylized models only arises from a modeling choice which allows the demand of commodities to be possibly greater than the total amount of the commodity that can be ever supplied. However, it is indeed possible to restrict the excess demand to bounded by the total amount of the commodity that can be ever supplied without modifying the Walrasian equilibria of the economy. This is indeed the approach that Arrow and Debreu (1954) take in Section 3 of their paper for proving their seminal Walrasian equilibrium existence result, and it is also the approach we will take in Chapter 10 to prove convergence of price adjustment processes in Arrow-Debreu economies. This restriction is also realistic from an economic perspective since it is not possible for the economy to consume more of a commodity that there can exist, and resources in the real-world are indeed scarce. Indeed, otherwise there would be no use for the economic sciences: the science of resource allocation under scarcity.

With the above theorem in hand, we can leverage the fact that a strong solution is guaranteed in continuous VIs (Theorem 4.1.1) to establish the existence of a Walrasian equilibrium in continuous competitive economies.

Theorem 5.2.4.
The set of Walrasian equilibria of any continuous competitive economy is non-empty.

Proof of Theorem 5.2.4

By Theorem 5.2.3, we know that the set of strong solutions $\mathcal{SVI}(\Delta_m, -\mathcal{Z})$ of the VI $(\Delta_m, -\mathcal{Z})$ is a subset of the set of Walrasian equilibria $(m, \mathcal{Z})$.
Now, notice that for a continuous Walrasian economy $(\Delta_m, -\mathcal{Z})$ is a continuous VI. Hence, by Theorem 4.1.1 a strong solution to $(\Delta_m, -\mathcal{Z})$ is guaranteed to exist, which in turn implies the existence of a Walrasian equilibrium in continuous competitive Walrasian economies.

5.3 Algorithms for Walrasian Equilibrium

5.3.1 Computational Model

To analyze the computational properties of the algorithms we will introduce, we will take two approaches. For balanced economies, we will introduce algorithms with polynomial-time global convergence guarantees to a Walrasian equilibrium. For general Walrasian economies, as by Theorem 5.2.1 the computation of a Walrasian equilibrium is equivalent the PPAD-hard problem of solving a complementarity problem (see, for instance, (IEOR, 2011)) a polynomial-time global convergence to a Walrasian equilibrium is likely to be impossible, and as such we will instead define a merit function (i.e., a function whose set of minimizers coincide with Walrasian equilibria), and provide polynomial-time convergence to approximate stationary points of this merit function (as defined in Chapter 4).

We will for the rest of this chapter, assume that the excess demand correspondence is singleton-valued unless otherwise noted. Similar to Chapter 4, we will consider two classes of methods to compute a Walrasian equilibrium, (first-order) price-adjustment processes and second-order price-adjustment processes which both belong to the class of kth order price adjustment processes.

Definition 5.3.1 [kth-order price-adjustment process]**.**
Given some $k \in \mathbb{N}_{++}$, and a Walrasian economy $(m, \mathcal{Z})$ for which the derivatives $\{\nabla^j \boldsymbol{z}\}_{j=1}^{k-1}$ are well defined, and an initial iterate $\boldsymbol{p}^{(0)} \in \mathbb{R}_+^m$, a **$k$th-order price adjustment process $\boldsymbol{\pi}$** consists of an update function which generates the sequence of iterates $\{\boldsymbol{p}^{(t)}\}_t$ given for all $t = 0, 1, \ldots$ by:

$$\boldsymbol{x}^{(t+1)} \doteq \boldsymbol{\pi}\left(\bigcup_{i=0}^{t}(\boldsymbol{p}^{(i)}, \{\nabla^j \boldsymbol{z}(\boldsymbol{p}^{(i)})\}_{j=0}^{k-1})\right)$$

The computational complexity results in this chapter will rely on the following computational model which has been broadly adopted by the literature (see, for instance, Papadimitriou and Yannakakis (2010)).

Definition 5.3.2 [Walrasian Computational Model]**.**
Given a Walrasian economy $(m, \mathcal{Z})$, and a kth-order price adjustment process $\boldsymbol{\pi}$, the computational complexity of $\boldsymbol{\pi}$ is measured in term of the number of evaluations of the the functions $\boldsymbol{z}, \nabla \boldsymbol{z}, \ldots, \nabla^k \boldsymbol{z}$.

5.3.2 Related Works

We review here some of the relevant literature in computer science, on price-adjustment processes for Walrasian equilibrium computation. This literature has in part been motivated by applications of algorithms such as *tâtonnement* to load balancing over networks (Jain et al., 2013), or to pricing of transactions on crypotocurrency blockchains (Leonardos et al., 2021; Liu et al., 2022; Reijsbergen et al., 2021). A detailed inquiry into the computational properties of Walrasian equilibria was initiated by Devanur et al. (2008), who studied a special case of the Arrow-Debreu competitive economy known as the **Fisher market** (Brainard et al., 2000). This model, for which Irving Fisher computed equilibrium prices using a hydraulic machine in the 1890s, is essentially the Arrow-Debreu model of a competitive economy, but there are no firms, and buyers are endowed with only one type of commodity—hereafter good[3]—an artificial currency (Brainard et al., 2000; Nisan and Roughgarden, 2007). Devanur et al. (2002) exploited a connection first made by Eisenberg (1961) between the **Eisenberg-Gale program** and competitive equilibrium to solve Fisher markets assuming buyers with linear utility functions, thereby providing a (centralized) polynomial-time algorithm for equilibrium computation in these markets (Devanur et al., 2002; Devanur et al., 2008). Their work was built upon by Jain et al. (2005), who extended the Eisenberg-Gale program to all Fisher markets in which buyers have **continuous, quasi-concave, and homogeneous** utility functions, and proved that the equilibrium of Fisher markets with such buyers can be computed in polynomial time by interior point methods.

Concurrent with this line of work on computing competitive equilibrium using centralized methods, a line of work on devising and proving convergence guarantees for price-adjustment processes (i.e., iterative algorithms that update prices according to a predetermined update rule) developed. This literature has focused on devising *natural* price-adjustment processes, like *tâtonnement*, which might explain or imitate the movement of prices in real-world markets. In addition to imitating the law of supply and demand, *tâtonnement* has been observed to replicate the movement of prices in lab experiments, where participants are given endowments and asked to trade with one another (Gillen et al., 2020). Perhaps more importantly, the main premise of research on the stability of competitive equilibrium in computer science is that for competitive equilibrium to be justified, not only should it be backed by a natural price-adjustment process as economists have long argued, but it should also be computationally efficient (Nisan and Roughgarden, 2007).

The first result on this question is due to Codenotti et al. (2005), who introduced a discrete-time version of *tâtonnement*, and showed that in exchange economies that satisfy **weak gross substitutes (WGS)**, the *tâtonnement* process converges to an approximate competitive equilibrium in a number of steps which is polynomial in the approximation factor and size of the problem. Unfortunately, soon after this positive result appeared, Papadimitriou and Yannakakis (2010) showed that it is impossible for a price-adjustment

[3] In the context of Fisher markets, commodities are typically referred to as goods (Cheung et al., 2013), as Fisher markets are often analyzed for a single time period only. More generally, in Arrow-Debreu markets, where commodities vary by time, location, or state of the world, "an apple today" may be different than "an apple tomorrow". For consistency with the literature, we refer to commodities as goods.

process based on the excess demand function to converge in polynomial time to a competitive equilibrium in general, ruling out the possibility of Smale's process (and many others) justifying the notion of competitive equilibrium in all competitive economies. Nevertheless, further study of the convergence of price-adjustment processes such as *tâtonnement* under stronger assumptions, or in simpler models than full-blown Arrow-Debreu competitive economies, remains worthwhile, as these processes are being deployed in practice (Jain et al., 2013; Leonardos et al., 2021; Liu et al., 2022; Reijsbergen et al., 2021).

Following Codenotti et al.'s [2005] initial analysis of *tâtonnement* in competitive economies that satisfy WGS, Garg and Kapoor (2004) introduced an auction algorithm that also converges in polynomial time for linear exchange economies. More recently, Bei et al. (2015) established faster convergence bounds for *tâtonnement* in WGS exchange economies.

Another line of work considers price-adjustment processes in variants of Fisher markets. Cole and Fleischer (2008) analyzed *tâtonnement* in a real-world-like model satisfying WGS called the ongoing market model. In this model, *tâtonnement* once-again converges in polynomial-time (Cole and Fleischer, 2008; Cole et al., 2010), and it has the advantage that it can be seen as an abstraction for market processes. Cole and Fleischer's results were later extended by Cheung et al. (2012) to ongoing markets with **weak gross complements**, i.e., the excess demand of any commodity weakly increases if the price of any other commodity weakly decreases, fixing all other prices, and ongoing markets with a mix of WGC and WGS commodities. The ongoing market model these two papers study contains as a special case the Fisher market; however Cole and Fleischer (2008) assume bounded own-price elasticity of Marshallian demand, and bounded income elasticity of Marshallian demand, while Cheung et al. (2012) assume, in addition to Cole and Fleischer's assumptions, bounded adversarial market elasticity, which can be seen as a variant of bounded cross-price elasticity of Marshallian demand, from below. With these assumptions, these results cover Fisher markets with a small range of the well-known CES utilities, including CES Fisher markets with $\rho \in [0, 1)$ and WGC Fisher markets with $\rho \in (-1, 0]$.

Cheung et al. (2013) built on this work by establishing the convergence of *tâtonnement* in polynomial time in nested CES Fisher markets, excluding the limiting cases of linear and Leontief markets, but nonetheless extending polynomial-time convergence guarantees for *tâtonnement* to Leontief Fisher markets as well. More recently, Cheung and Cole (2018) showed that Cheung et al.'s [2013] result extends to an asynchronous version of *tâtonnement*, in which good prices are updated during different time periods. In a similar vein, Cheung et al. (2019) analyzed *tâtonnement* in online Fisher markets, determining that *tâtonnement* tracks competitive equilibrium prices closely provided the market changes slowly.

Another price-adjustment process that has been shown to converge to Walrasian equilibrium in Fisher markets is **proportional response dynamics**, first introduced by Wu and Zhang (2007) for linear utilities; then expanded upon and shown to converge by (Zhang, 2011) for all CES utilities; and very recently shown to converge in Arrow-Debreu exchange economies with linear and CES ($\rho \in [0, 1)$) utilities by Brânzei et al.. The study of the proportional response process was proven fundamental when Cheung et al. noticed its relationship to gradient descent. This discovery opened up a new realm of possibilities in analyzing the convergence of Walrasian equilibrium processes. For example, it allowed Cheung et al. (2018) to generalize the convergence results of proportional response dynamics to Fisher markets for buyers with mixed CES utilities. This same idea was applied by Cheung et al. (2013) to prove the convergence of *tâtonnement* in Leontief Fisher markets, using the equivalence between mirror descent (Boyd et al., 2004) on the dual of the Eisenberg-Gale program and *tâtonnement*, first observed by Devanur et al. (2008). More recently, Gao and Kroer (2020) developed methods to solve the Eisenberg-Gale convex program in the case of linear, quasi-linear, and Leontief Fisher markets.

An alternative to the (global) competitive economy model, in which an agent's trading partners are unconstrained, is the Kakade et al. (2004) model of a graphical economies. This model features local markets, in which each agent can set its own prices for purchase only by neighboring agents, and likewise can purchase only from neighboring agents. Auction-like price-adjustment processes have been shown to converge in variants of this model assuming WGS (Andrade et al., 2021).

5.4 Price Adjustment Processes for Walrasian Equilibrium

The most common class of algorithms to compute a Walrasian equilibrium are first-order price adjustment processes simply called **price adjustment processes** (Papadimitriou and Yannakakis, 2010).

Definition 5.4.1 [Price-adjustment process]**.**

Given a Walrasian economy $(m, \boldsymbol{z})$, and an initial price vector $\boldsymbol{p}^{(0)} \in \mathbb{R}^m_+$, a **price adjustment process** $\boldsymbol{\pi}$ consists of an update function $\boldsymbol{\pi} : \bigcup_{\tau \geq 1}(\mathbb{R}^m_+ \times \mathbb{R}^m) \to \mathbb{R}^m_+$ which generates the sequence of prices $\{\boldsymbol{p}^{(t)}\}_t$ given for all $t = 0, 1, \dots$ by:

$$\boldsymbol{p}^{(t+1)} \doteq \boldsymbol{\pi}\left(\bigcup_{k=0}^{t}(\boldsymbol{p}^{(k)}, \boldsymbol{z}(\boldsymbol{p}^{(k)}))\right)$$

An important class of price-adjustment processes are natural price-adjustment processes. Intuitively, these are price-adjustment processes where the price of each commodity is updated using only information about the past prices of the commodity and its excess demand. This class of processes are natural in the sense that the price of each commodity is updated with information relevant to it, and as such if each commodity is sold by one fictional seller, then each seller can update the price of its good without having to coordinate with other seller.

Definition 5.4.2 [Natural Price-Adjustment Process]**.**

Given a Walrasian economy $(m, \boldsymbol{z})$, and an initial price vector $\boldsymbol{p}^{(0)} \in \mathbb{R}^m_+$, a price adjustment process $\boldsymbol{\pi}$ is said to be **natural** if for all commodities, the price adjustment process can be written as as $\boldsymbol{\pi} \doteq (\pi_1, \dots, \pi_m)$ where for all commodities $j \in [m]$, $\pi_j : \bigcup_{\tau \geq 1}(\mathbb{R}_+ \times \mathbb{R}) \to \mathbb{R}^m_+$ s.t. for all $t = 0, 1, \dots$ we have:

$$p_j^{(t+1)} \doteq \pi_j\left(\bigcup_{k=0}^{t}(p_j^{(k)}, z_j(\boldsymbol{p}^{(k)}))\right)$$

The canonical type of natural price adjustment processes are *tâtonnement processes* (Walras, 1896; Arrow and Hurwicz, 1958).

Definition 5.4.3 [*Tâtonnement* process]**.**

A *tâtonnement process* is a natural price adjustment process $\boldsymbol{\pi} \doteq (\pi_1, \dots, \pi_m)$ s.t. for all $j \in [m]$ and $t \in \mathbb{N}_{++}$ there exists a function $g : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}$ that satisfies:

$$\pi_j\left(\bigcup_{k=0}^{t}(p_j^{(k)}, z_j(\boldsymbol{p}^{(k)}))\right) \doteq g(p_j^{(t)}, z_j(\boldsymbol{p}^{(t)})) \tag{5.9}$$

Remark 5.4.1 [On *tâtonnement*]**.**

The verb *tâtonner* in French means to search by trial and error, often connoting a sense of blindness, as the search relies solely on local information. Accordingly, the noun form *tâtonnement* describes a heuristic search process based on trial and error. The *tâtonnement* process is a memoryless price adjustment mechanism where each commodity's next price is determined solely by its current price and excess demand. The term *tâtonnement* is thus aptly chosen, as the price search for each commodity is heuristic, ignoring both past prices and excess demands as well as the current prices and excess demands of other goods.

Traditionally, g is further restricted to be sign preserving, i.e., $\forall p \in \mathbb{R}_+, z \in \mathbb{R}, \text{sign}(g(p, z)) = \text{sign}(z)$, as with this restriction in place a *tâtonnement* process can be seen a mathematical model of the law supply and demand which stipulates that the price of any commodity in the economy which is demanded (resp. supplied) in excess will rise (resp. decrease) Walras (1896); Arrow and Hurwicz (1958).

Now notice that the mirror gradient method applied to the VI $(\mathbb{R}^m_+, -\mathcal{Z})$ defines a family of *tâtonnement* processes parametrized by the kernel function h which we will call the mirror *tâtonnement* process. Unfortunately, while continuous-time variants of the *tâtonnement processes* are known to converge in Arrow-Debreu economies for which the excess demand $\boldsymbol{z}$ is for instance monotone (in which case the excess demand is said to satisfy the **law of demand**, see Definition 5.4.8), as Example 4.3.1 shows the mirror *tâtonnement* process is not guaranteed to converge in such economies.[4] Nevertheless, recall that we can instead apply the mirror extragradient algorithm to the VI $(\mathbb{R}^m_+, -\mathcal{Z})$, which as we have shown earlier can be guaranteed to converge in VIs which satisfy the Minty condition using the tools developed in Chapter 4.

Definition 5.4.4 [Variationally Stable Walrasian Economies]**.**

A Walrasian economy $(m, \mathcal{Z})$, is said to be **variationally stable** on $\mathcal{P} \subseteq \mathbb{R}^m_+$ iff there exists $\boldsymbol{p}^* \in \mathcal{P}$ s.t. for all prices $\boldsymbol{p} \in \mathcal{P}$, $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p})$:

$$\langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p}^* - \boldsymbol{p} \rangle \geq 0$$

If a balanced economy is variationally stable on Δ_m, then we refer to the economy simply as **variationally stable**.

[4]While this example is presented for VIs, by the equivalence between VIs and Walrasian economies, it also applies to Walrasian economies.

To understand the variational stability condition, consider a fictional auctioneer who buys the commodities sold in the economy and sells them back at prices $\boldsymbol{p} \in \mathbb{R}_+^m$. The profit of the auctioneer for his transaction is given by $\langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p} \rangle$. Now suppose that the auctioneer where to change the prices at which it bought and sold its commodities to prices $\boldsymbol{p}^* \in \mathbb{R}_+^m$, while fixing the quantities of goods sold and bought to those that he has observed (i.e., the excess demand $\boldsymbol{z}(\boldsymbol{p})$), then the auctioneer's profit improvement in "hindsight" would be given by $\langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p}^* \rangle - \langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p}^* \rangle = \langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p}^* - \boldsymbol{p} \rangle$. Then, the Minty condition requires the existence of a price vector $\boldsymbol{p}^* \in \mathbb{R}_+^m$, which in hindsight looks to the auctioneer like a more profitable price vector than any original price vector $\boldsymbol{p} \in \mathbb{R}_+^m$ it chose.

5.4.1 Computation of Walrasian Equilibrium in Balanced Economies

A surprising and important result which is described by the following lemma is that the VI $([0,1]^m, -\mathcal{Z})$ associated with any balanced economy $(m, \mathcal{Z})$ satisfies the Minty condition, i.e., any balanced economy is variationally stable on $[0,1]^m$.

Lemma 5.4.1 [Balanced Economies are Variationally Stable]**.**
Any balanced economy $(m, \mathcal{Z})$ is variationally stable on $[0,1]^m$, in particular letting $\boldsymbol{p}^* \doteq \boldsymbol{0}_m$, for all prices $\boldsymbol{p} \in [0,1]^m$ and $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p})$, we have:

$$\langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p}^* - \boldsymbol{p} \rangle \geq 0$$

Proof of Lemma 5.4.1

Let $(m, \mathcal{Z})$ be a balanced economy, then setting $\boldsymbol{p}^* \doteq \boldsymbol{0}_m$, we have:

$$\begin{aligned}
\langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p}^* - \boldsymbol{p} \rangle &= \langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{0}_m - \boldsymbol{p} \rangle \\
&= \underbrace{\langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{0}_m \rangle}_{=0} - \langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p} \rangle \\
&= -\underbrace{\langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p} \rangle}_{\leq 0} \\
&\geq 0
\end{aligned}$$

where the last line follow from the weak Walras' law assumption holding in balanced economies, i.e., for all prices $\boldsymbol{p} \in [0,1]^m$, $\langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p} \rangle \leq 0$, which implies $-\langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p} \rangle \geq 0$.

The above lemma is a highly surprising and important as it suggests that for balanced economies which include among others Arrow-Debreu economies (see Chapter 10 for additional details) under suitable continuity assumptions, first-order methods for the VI $([0,1]^m, -\mathcal{Z})$ are guaranteed to converge to a strong solution.

Hence, with Lemma 5.4.1 in hand, we now turn our attention to solving the VI $([0,1]^m, -\mathcal{Z})$—or in our case rather the VI $([0,1]^m, -\boldsymbol{z})$ since we assume for our algorithms that the excess demand is singleton-valued—and hence a Walrasian equilibrium with the mirror extragradient method. Solving the VI $([0,1]^m, -\boldsymbol{z})$ with the mirror extragradient method, gives rise to a family of price adjustment processes parameterized by the kernel function h which we will call the **mirror extratâtonnement process**.

Algorithm 6 Mirror Extratâtonnement Process

Input: $m, \boldsymbol{z}, \tau, \eta, h, \mathcal{P}, \boldsymbol{p}^{(0)}$

Output: $\{\boldsymbol{p}^{(t)}\}_{t \in [\tau]}$

1: **for** $t = 1, \dots, \tau$ **do**

2: $\quad \boldsymbol{p}^{(t+0.5)} \leftarrow \arg\min_{\boldsymbol{p} \in \mathcal{P}} \left\{ \left\langle \boldsymbol{z}(\boldsymbol{p}^{(t)}), \boldsymbol{p}^{(t)} - \boldsymbol{p} \right\rangle + \frac{1}{2\eta} \mathrm{div}_h(\boldsymbol{p}, \boldsymbol{p}^{(t)}) \right\}$

3: $\quad \boldsymbol{p}^{(t+1)} \leftarrow \arg\min_{\boldsymbol{p} \in \mathcal{P}} \left\{ \left\langle \boldsymbol{z}(\boldsymbol{p}^{(t+0.5)}), \boldsymbol{p}^{(t)} - \boldsymbol{p} \right\rangle + \frac{1}{2\eta} \mathrm{div}_h(\boldsymbol{p}, \boldsymbol{p}^{(t)}) \right\}$

return $\{\boldsymbol{p}^{(t+0.5)}\}_{t \in [\tau]}$

Remark 5.4.2 [Mirror *Extratâtonnement* is a Natural Price-Adjustment Process]**.**

For the choice of a price space $\mathcal{P} \doteq [0,1]^m$, and any choice of kernel function s.t. $h(\boldsymbol{p}) \doteq \sum_{j[m]} h_j(p_j)$ for some $\{h_j : \mathbb{R}^m \to \mathbb{R}\}_{j \in [m]}$, the mirror *extratâtonnement* updates can be written for all commodities $j \in [m]$ and $t \in \mathbb{N}$ as:

$$p_j^{(t+0.5)} \leftarrow \arg\min_{p_j \in [0,1]} \left\{ \left\langle z_j(\boldsymbol{p}^{(t)}), p_j^{(t)} - p_j \right\rangle + \frac{1}{2\eta} \mathrm{div}_{h_j}(p_j, p_j^{(t)}) \right\}$$
$$p_j^{(t+1)} \leftarrow \arg\min_{p_j \in [0,1]} \left\{ \left\langle z_j(\boldsymbol{p}^{(t+0.5)}), p_j^{(t)} - p_j \right\rangle + \frac{1}{2\eta} \mathrm{div}_{h_j}(p_j, p_j^{(t)}) \right\} \quad .$$

Further, multiplying the index of the sequence of price iterates, we can re-write the above update rule for all commodities $j \in [m]$ and $t \in \mathbb{N}$ as:

$$p_j^{(t+1)} \leftarrow \underset{p_j \in [0,1]}{\arg\min} \left\{ \left\langle z_j(\boldsymbol{p}^{(t)}), p_j^{(t)} - p_j \right\rangle + \frac{1}{2\eta} \mathrm{div}_{h_j}(p_j, p_j^{(1)}) \right\}$$
$$p_j^{(t+2)} \leftarrow \underset{p_j \in [0,1]}{\arg\min} \left\{ \left\langle z_j(\boldsymbol{p}^{(t+1)}), p_j^{(t)} - p_j \right\rangle + \frac{1}{2\eta} \mathrm{div}_{h_j}(p_j, p_j^{(t)}) \right\} \quad .$$

That is, the mirror *extratâtonnement* process (Algorithm 6) can be interpretted as a natural price adjustment process which on odd time-steps applies a *tâtonnement* update on the current time-step's prices, and on even time-step applies a *tâtonnement* update on the *previous* time-step's prices. As such, the mirror *extratâtonnement* process is natural price adjustment process.

With the mirror *extrâtonnement* process, and Lemma 5.4.1 in hand, we can apply Theorem 4.3.1 to prove the convergence of the mirror *extrâtonnement* process (Algorithm 6).

Theorem 5.4.1 [Convergence of Mirror Extratâtonnement]**.**
Let $(m, \boldsymbol{z})$ be a balanced economy. Consider the mirror *extrâtonnement* process run on $(m, \boldsymbol{z})$, with a 1-strongly-convex and κ-Lipschitz-smooth kernel function h, any time horizon $t \doteq \mathbb{N}$, any step size $\eta > 0$, a price space $\mathcal{P} \doteq [0, 1]^m$, and any initial price vector $\boldsymbol{p}^{(0)} \in [0, 1]^m$, and let $\{\boldsymbol{p}^{(t)}, \boldsymbol{p}^{(t+0.5)}\}_t$ be the sequence of prices generated. Suppose that there exists $\lambda \in (0, \frac{1}{\sqrt{2\eta}}]$, s.t. $\frac{1}{2}\|\boldsymbol{z}(\boldsymbol{p}^{(k+0.5)}) - \boldsymbol{z}(\boldsymbol{p}^{(k)})\|^2 \leq \lambda^2 \mathrm{div}_h(\boldsymbol{p}^{(k+0.5)}, \boldsymbol{p}^{(k)})$. Let $\boldsymbol{p}_{\text{best}}^{(\tau)} \in \arg\min_{\boldsymbol{x}^{(k+0.5)}:k=0,\ldots,\tau} \mathrm{div}_h(\boldsymbol{p}^{(k+0.5)}, \boldsymbol{p}^{(k)})$, then for some time horizon $\tau \doteq O(\frac{\kappa^2 m^2 \mathrm{div}_h(\boldsymbol{0}_m, \boldsymbol{p}^{(0)})}{\eta^2 \varepsilon^2})$, $\boldsymbol{p}_{\text{best}}^{(\tau)}$ is a ε-Walrasian equilibrium.

Further, we have that $\lim_{t\to\infty} \boldsymbol{p}^{(t+0.5)} = \lim_{t\to\infty} \boldsymbol{p}^{(t)} = \boldsymbol{p}^*$ is a Walrasian equilibrium.

Proof of Theorem 5.4.1

Since $(m, \boldsymbol{z})$ is a balanced economy, by Lemma 5.4.1, $(m, \boldsymbol{z})$ is variationally stable on $[0, 1]^m$, and hence the VI $([0, 1]^m, -\mathcal{Z})$ satisfies the Minty condition. Hence, as the mirror *extratâtonnement* process is simply the mirror extragradient method run on the VI $([0, 1]^m, -\mathcal{Z})$, the assumptions of Theorem 4.3.1 are satisfied, and we obtain the result.

The convergence guarantee provided by the above theorem is highly general, and does not require Lipschitz-continuity of the excess demand $\boldsymbol{z}$. Rather, the above theorem requires a notion "Bregman-continuity over trajectories" of the *extratâtonnement* process. This broad statement is purposeful as it is in general not possible to guarantee the Lipschitz-continuity of the excess demand in balanced economies. Indeed, the only balanced economies with a Lipschitz-continuous excess demand function are those economies with a constant excess demand function. To see this, suppose that $\boldsymbol{z}$ is λ-Lipschitz-continuous on $[0,1]^m$, then by homogeneity of degree 0, we have for all $\alpha > 0$ and $\boldsymbol{p}, \boldsymbol{q} \in [0,1]^m$:

$$\|\boldsymbol{z}(\boldsymbol{p}) - \boldsymbol{z}(\boldsymbol{q})\| = \|\boldsymbol{z}(\alpha\boldsymbol{p}) - \boldsymbol{z}(\alpha\boldsymbol{q})\| \leq \lambda\alpha\|\boldsymbol{p} - \boldsymbol{q}\| \tag{5.10}$$

Hence, taking $\alpha \to 0$, we have for all $\boldsymbol{p}, \boldsymbol{q} \in [0,1]^m$, $\boldsymbol{z}(\boldsymbol{q}) = \boldsymbol{z}(\boldsymbol{p})$. Nevertheless, while Lipschitz-continuity over $[0,1]^m$ is too restrictive, Lipschitz continuity over paths of the mirror *extratâtonnement* process which we call **pathwise Lipschitz-continuity** (i.e., $\|\boldsymbol{z}(\boldsymbol{p}^{(k+0.5)}) - \boldsymbol{z}(\boldsymbol{p}^{(k)})\| \leq \lambda \|\boldsymbol{p}^{(k+0.5)} - \boldsymbol{p}^{(k)}]\|$)[5] seems to be a mild assumption that holds in a large class of Walrasian economies as the experiments in Section 5.4.3. Further, it seems likely that for choices of kernel functions h s.t. the associated Bregman divergence div_h is not homogeneous of degree $\alpha > 0$ (i.e., for all $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^m_+, \alpha, \lambda > 0$ $\mathrm{div}_h(\lambda\boldsymbol{p}, \lambda\boldsymbol{q}) \neq \lambda^\alpha \mathrm{div}_h(\boldsymbol{p}, \boldsymbol{q})$), the following class of Walrasian economies seems to be likely to contain a large number of Walrasian economies.

Definition 5.4.5 [Bregman-continuous economies]**.**
Given $\lambda \geq 0$, and a kernel function $h : \mathcal{P} \to \mathbb{R}$, a (λ, h)**-Bregman-continuous economy** on $\mathcal{P} \subseteq \mathbb{R}^m_+$ is a Walrasian economy $(m, \mathcal{Z})$ whose excess demand correspondence is singleton-valued (i.e., $\mathcal{Z}(\boldsymbol{p}) \doteq \{\boldsymbol{z}(\boldsymbol{p})\}$) and (λ, h)**-Bregman-continuous** on $\mathcal{P}$, i.e., for all $\boldsymbol{p}, \boldsymbol{q} \in \mathcal{P}$:

$$\frac{1}{2}\|\boldsymbol{z}(\boldsymbol{p}) - \boldsymbol{z}(\boldsymbol{q})\|^2 \leq \lambda^2 \mathrm{div}_h(\boldsymbol{q}, \boldsymbol{p})$$

When h is clear from context, we simply say that the economy is λ-Bregman continuous on $\mathcal{P}$ and $\boldsymbol{z}$ is λ-Bregman continuous on $\mathcal{P}$.

Bregman continuous functions have been introduced in recent years in the optimization literature and have been shown to contain a large number of important function classes which are not continuous (see, for instance, Lu (2019)). Note that when the kernel function h is chosen to be $h(\boldsymbol{p}) \doteq \frac{1}{2}\|\boldsymbol{p}\|^2$, λ-Bregman-continuity reduces to λ-Lipschitz continuity. Further, the literature on algorithmic general equilibrium theory has considered variants of Bregman continuity to prove the polynomial-time convergence of algorithms

[5] Notice that when the kernel function in the statement of Theorem 5.4.1 is chosen to be $h(\boldsymbol{p}) \doteq \frac{1}{2}\|\boldsymbol{p}\|^2$, the condition $\frac{1}{2}\|\boldsymbol{z}(\boldsymbol{p}^{(k+0.5)}) - \boldsymbol{z}(\boldsymbol{p}^{(k)})\|^2 \leq \lambda^2 \mathrm{div}_h(\boldsymbol{p}^{(k+0.5)}, \boldsymbol{p}^{(k)})$ reduces to $\|\boldsymbol{z}(\boldsymbol{p}^{(k+0.5)}) - \boldsymbol{z}(\boldsymbol{p}^{(k)})\| \leq \lambda\|\boldsymbol{p}^{(k+0.5)} - \boldsymbol{p}^{(k)}]\|$.

to Walrasian equilibria (see, for instance Cheung et al. (2013) and Cheung et al. (2018)). As such, Bregman continuity seems a natural assumption to prove the convergence of algorithms to a Walrasian equilibrium.

With the definition of Bregman continuous economies in hand, we obtain the following corollary of Theorem 5.4.1.

Corollary 5.4.1.

Let $(m, \boldsymbol{z})$ be a balanced economy which is (λ, h)-Lipschitz-continuous on $[0,1]^m$. Consider the mirror *extrâtonnement* process run on $(m, \boldsymbol{z})$, with a 1-strongly-convex and κ-Lipschitz-smooth kernel function h, any time horizon $t \doteq \mathbb{N}$, any step size $\eta \in (0, \frac{1}{\sqrt{2}\lambda}]$, a price space $\mathcal{P} \doteq [0,1]^m$, and any initial price vector $\boldsymbol{p}^{(0)} \in [0,1]^m$, and let $\{\boldsymbol{p}^{(t)}, \boldsymbol{p}^{(t+0.5)}\}_t$ be the sequence of prices generated.

Let $\boldsymbol{p}^{(\tau)}_{\text{best}} \in \arg\min_{\boldsymbol{x}^{(k+0.5)}:k=0,\dots,\tau} \text{div}_h(\boldsymbol{p}^{(k+0.5)}, \boldsymbol{p}^{(k)})$, then for some time horizon $\tau \doteq O(\frac{\kappa^2 m^2 \text{div}_h(\boldsymbol{0}_m, \boldsymbol{p}^{(0)})}{\eta^2 \varepsilon^2})$, $\boldsymbol{p}^{(\tau)}_{\text{best}}$ is a ε-Walrasian equilibrium. Further, we have that $\lim_{t\to\infty} \boldsymbol{p}^{(t+0.5)} = \lim_{t\to\infty} \boldsymbol{p}^{(t)} = \boldsymbol{p}^*$ is a Walrasian equilibrium.

While these convergences result are useful, it is not clear what types of excess demand functions satisfy Bregman-continuity. As a result, to characterize the Bregman-continuity properties of Walrasian economies we introduce the following economic parameters which have been extensively used in the analysis of algorithms for the computation of Walrasian equilibrium (see, for instance, Cole and Fleischer (2008)).

Definition 5.4.6 [Function Elasticity]**.**

Given any function $\boldsymbol{f} : \mathbb{R}^n \to \mathbb{R}^m$, we define the **elasticity** $\epsilon_{f_j, p_k} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of output f_j w.r.t. input x_k between any two points $\boldsymbol{x} \in \mathbb{R}^n$ and $\boldsymbol{y} \in \mathbb{R}^n$ as the percentage change in f_j for a one percent change from x_k to y_k:

$$\epsilon_{f_j, x_k}(\boldsymbol{x}, \boldsymbol{y}) \doteq \frac{f_j(\boldsymbol{y}) - f_j(\boldsymbol{x})}{f_j(\boldsymbol{x})} \frac{x_k}{y_k - x_k} \tag{5.11}$$

Overloading notation, we also define the **instantaneous elasticity** as follows:

$$\epsilon_{f_j, x_k}(\boldsymbol{x}, \boldsymbol{y}) \doteq \lim_{h\to 0} \frac{\epsilon_{f_j, x_k}(\boldsymbol{x}, \boldsymbol{x} + h\boldsymbol{j}_k)}{h} = \frac{\partial_{x_k} f_j(\boldsymbol{x}) x_k}{f_j(\boldsymbol{x})} \tag{5.12}$$

Definition 5.4.7 [Elastic economies]**.**

Given $\bar{\epsilon} \geq 0$, a $\bar{\epsilon}$**-elastic economy** $(m, \boldsymbol{d}, \boldsymbol{s})$ is a Walrasian economy $(m, \mathcal{Z})$ which consists of an **aggregate demand function** $\boldsymbol{d} : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ and **aggregate supply function** $\boldsymbol{s} : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ s.t. we have $\mathcal{Z}(\boldsymbol{p}) \doteq$

$\{\boldsymbol{d}(\boldsymbol{p})\} - \{\boldsymbol{s}(\boldsymbol{p})\}$, and the following two bounds hold:

$$\max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\mathbb{R}^m_+\\ j,k\in[m]}} \left|\epsilon_{d_j,p_k}(\boldsymbol{p},\boldsymbol{q})\right| \leq \bar{\epsilon}, \qquad \max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\mathbb{R}^m_+\\ j,k\in[m]}} \left|\epsilon_{s_j,p_k}(\boldsymbol{p},\boldsymbol{q})\right| \leq \bar{\epsilon}$$

The following lemma demonstrates that the excess demand of any $\bar{\epsilon}$-economy with a bounded aggregate demand and aggregate supply is Bregman-continuous w.r.t to the log-barrier kernel $h(\boldsymbol{p}) = \sum_{j\in[m]} - \log(p_j)$.

Lemma 5.4.2 [Bregman Continuity for elastic economies]**.**

Let $(m, \boldsymbol{d}, \boldsymbol{s})$ be $\bar{\epsilon}$-elastic economy, then for any 1-strongly-convex kernel function $h : \mathbb{R}^m_+ \to \mathbb{R}$, the following bound holds:

$$\frac{1}{2}\|\boldsymbol{z}(\boldsymbol{q}) - \boldsymbol{z}(\boldsymbol{p})\|^2 \leq \left(\frac{\epsilon\left(\|\boldsymbol{d}(\boldsymbol{p})\| + \|\boldsymbol{s}(\boldsymbol{p})\|\right)}{\|\boldsymbol{p}\|_\infty}\right)^2 \mathrm{div}_h(\boldsymbol{q},\boldsymbol{p})$$

Proof of Lemma 5.4.2

By the assumption of the theorem, we have for all $\boldsymbol{p},\boldsymbol{q} \in \Delta_m$, $j,k \in [m]$:

$$\left|\frac{d_j(\boldsymbol{q}) - d_j(\boldsymbol{p})}{d_j(\boldsymbol{p})}\frac{p_k}{q_k - p_k}\right| \leq \epsilon$$

$$\frac{|d_j(\boldsymbol{q}) - d_j(\boldsymbol{p})|}{|d_j(\boldsymbol{p})|}\frac{|p_k|}{q_k - p_k} \leq \epsilon$$

$$|d_j(\boldsymbol{q}) - d_j(\boldsymbol{p})| \leq \frac{\epsilon|d_j(\boldsymbol{p})|}{p_k}|q_k - p_k|$$

$$|d_j(\boldsymbol{q}) - d_j(\boldsymbol{p})|^2 \leq \frac{\epsilon^2|d_j(\boldsymbol{p})|^2}{(p_k)^2}|q_k - p_k|^2$$

Summing up over $j \in [m]$, we have for all $k \in [m]$:

$$\begin{aligned}\|\boldsymbol{d}(\boldsymbol{q}) - \boldsymbol{d}(\boldsymbol{p})\|^2 &\leq \frac{\epsilon^2\|\boldsymbol{d}(\boldsymbol{p})\|^2}{(p_k)^2}|q_k - p_k|^2\\ &\leq \frac{\epsilon^2\|\boldsymbol{d}(\boldsymbol{p})\|^2}{(p_k)^2}\|\boldsymbol{q} - \boldsymbol{p}\|^2\end{aligned}$$

Since h is 1-strongly-convex, $\forall \boldsymbol{x},\boldsymbol{y} \in \mathcal{X}$, $\mathrm{div}_h(\boldsymbol{x},\boldsymbol{y}) \geq 1/2\|\boldsymbol{x} - \boldsymbol{y}\|^2$, hence, we have:

$$\|\boldsymbol{d}(\boldsymbol{q}) - \boldsymbol{d}(\boldsymbol{p})\|^2 \leq \frac{2\epsilon^2\|\boldsymbol{d}(\boldsymbol{p})\|^2}{(p_k)^2}\mathrm{div}_h(\boldsymbol{q},\boldsymbol{p})$$

Taking the square root of both sides and the taking a minimum over $k \in [m]$, we have:

$$\begin{aligned}\|\boldsymbol{d}(\boldsymbol{q}) - \boldsymbol{d}(\boldsymbol{p})\| &\leq \min_{k\in[m]}\frac{\epsilon\|\boldsymbol{d}(\boldsymbol{p})\|}{p_k}\sqrt{2\mathrm{div}_h(\boldsymbol{q},\boldsymbol{p})}\\ &= \frac{\epsilon\|\boldsymbol{d}(\boldsymbol{p})\|}{\max_{k\in[m]} p_k}\sqrt{2\mathrm{div}_h(\boldsymbol{q},\boldsymbol{p})}\\ &= \frac{\epsilon\|\boldsymbol{d}(\boldsymbol{p})\|}{\|\boldsymbol{p}\|_\infty}\sqrt{2\mathrm{div}_h(\boldsymbol{q},\boldsymbol{p})}\end{aligned}$$

By a similar argument, note that we also have:

$$\|\boldsymbol{s}(\boldsymbol{q}) - \boldsymbol{s}(\boldsymbol{p})\| \leq \frac{\epsilon\|\boldsymbol{s}(\boldsymbol{p})\|}{\|\boldsymbol{p}\|_\infty}\sqrt{2\mathrm{div}_h(\boldsymbol{q}, \boldsymbol{p})}$$

Combining the two above bounds, we then have:

$$\begin{aligned}
\|\boldsymbol{z}(\boldsymbol{q}) - \boldsymbol{z}(\boldsymbol{p})\| &= \|\boldsymbol{d}(\boldsymbol{q}) - \boldsymbol{s}(\boldsymbol{q}) - \boldsymbol{d}(\boldsymbol{p}) + \boldsymbol{s}(\boldsymbol{p})\| \\
&\leq \|\boldsymbol{d}(\boldsymbol{q}) - \boldsymbol{d}(\boldsymbol{p})\| + \|\boldsymbol{s}(\boldsymbol{q}) - \boldsymbol{s}(\boldsymbol{p})\| \\
&\leq \frac{\epsilon\|\boldsymbol{d}(\boldsymbol{p})\|}{\|\boldsymbol{p}\|_\infty}\sqrt{2\mathrm{div}_h(\boldsymbol{q}, \boldsymbol{p})} + \frac{\epsilon\|\boldsymbol{s}(\boldsymbol{p})\|}{\|\boldsymbol{p}\|_\infty}\sqrt{2\mathrm{div}_h(\boldsymbol{q}, \boldsymbol{p})} \\
&\leq \frac{\epsilon\|\boldsymbol{d}(\boldsymbol{p})\| + \|\boldsymbol{s}(\boldsymbol{p})\|}{\|\boldsymbol{p}\|_\infty}\sqrt{2\mathrm{div}_h(\boldsymbol{q}, \boldsymbol{p})}
\end{aligned}$$

Squaring both sides and re-organizing expressions, we then have:

$$\frac{1}{2}\|\boldsymbol{z}(\boldsymbol{q}) - \boldsymbol{z}(\boldsymbol{p})\|^2 \leq \left(\frac{\epsilon\left(\|\boldsymbol{d}(\boldsymbol{p})\| + \|\boldsymbol{s}(\boldsymbol{p})\|\right)}{\|\boldsymbol{p}\|_\infty}\right)^2 \mathrm{div}_h(\boldsymbol{q}, \boldsymbol{p})$$

Lemma 5.4.2 suggests that boundedness of the excess demand and a lower bound on the prices is sufficient to ensure the Bregman-continuity of the excess demand. Boundedness of the excess demand can be ensured in large class of Walrasian economies including Arrow-Debreu economies. While it is not possible to ensure that the prices are bounded from below, since Theorem 5.4.1 requires Bregman continuity over paths of the mirror extratâtonnement, we have the following corollary of Theorem 5.4.1.

Corollary 5.4.2 [Convergence of Mirror Extratâtonnement]**.**
Let $(m, \boldsymbol{d}, \boldsymbol{s})$ be a balanced and $\bar{\epsilon}$-elastic economy. Consider the mirror *extrâtonnement* process run on $(m, \boldsymbol{z})$, with a 1-strongly-convex and κ-Lipschitz-smooth kernel function h, any time horizon $t \doteq \mathbb{N}$, any step size $\eta > 0$, a price space $\mathcal{P} \doteq [0, 1]^m$, and any initial price vector $\boldsymbol{p}^{(0)} \in [0, 1]^m$, and let $\{\boldsymbol{p}^{(t)}, \boldsymbol{p}^{(t+0.5)}\}_t$ be the sequence of prices generated. Suppose that the step size satisfies $\eta \leq \min_{k \in [\tau]} \left\{\frac{\|\boldsymbol{p}^{(t)}\|_\infty}{\epsilon(\|\boldsymbol{d}(\boldsymbol{p}^{(t)})\| + \|\boldsymbol{s}(\boldsymbol{p}^{(t)})\|)}\right\}$.

Let $\boldsymbol{p}^{(\tau)}_{\text{best}} \in \arg\min_{\boldsymbol{x}^{(k+0.5)}: k=0,\ldots,\tau} \mathrm{div}_h(\boldsymbol{p}^{(k+0.5)}, \boldsymbol{p}^{(k)})$, then for some time horizon $\tau \doteq O(\frac{\kappa^2 m^2 \mathrm{div}_h(\boldsymbol{0}_m, \boldsymbol{p}^{(0)})}{\eta^2 \varepsilon^2})$, $\boldsymbol{p}^{(\tau)}_{\text{best}}$ is a ε-Walrasian equilibrium.

Further, if the step size satisfies $\eta \leq \min_{k \in \mathbb{N}} \left\{\frac{\|\boldsymbol{p}^{(t)}\|_\infty}{\epsilon(\|\boldsymbol{d}(\boldsymbol{p}^{(t)})\| + \|\boldsymbol{s}(\boldsymbol{p}^{(t)})\|)}\right\}$, we have $\lim_{t \to \infty} \boldsymbol{p}^{(t+0.5)} = \lim_{t \to \infty} \boldsymbol{p}^{(t)} = \boldsymbol{p}^*$ is a Walrasian equilibrium.

Unfortunately, beyond the above theorem we are unable to obtain a stronger polynomial-time convergence result in elastic balanced economies. It is possible that such a result requires a more fine grained analysis using a particular kernel function or perhaps some additional assumptions. We describe some possible directions for future work to obtain polynomial-time convergence in all elastic balanced economies, and provide an example of how stronger convergence results using a more fine grained analysis can be obtained using different kernel functions in Chapter 6.

Remark 5.4.3 [Directions for future work]**.**

For balanced economies, one suitable choice of kernel function which future work should look at is the **logistic loss** $h_{\text{LL}}(\boldsymbol{p}) \doteq \sum_{j\in[m]} \left[p_j \log(p_j) + (1-p_j)\log(1-p_j)\right]$ defined on $\mathcal{P} \doteq [0,1]^m$, which defines the **logistic divergence** $\text{div}_{h_{\text{LL}}}(\boldsymbol{p},\boldsymbol{q}) \doteq \sum_{j\in[m]} \left[p_j \log\left(\frac{p_j}{q_j}\right) + (1-p_j)\log\left(\frac{1-p_j}{1-q_j}\right)\right]$. For this choice of a kernel function, the mirror **extratâtonnement** process reduces to the **logistic extratâtonnement process** defined as for all commodities $j \in [m]$ and time horizons $t \in \mathbb{N}$:

$$p_j^{(t+0.5)} \doteq \frac{1}{1 + \frac{1-p_j^{(t)}}{p_j^{(t)}} e^{-\eta z_j(\boldsymbol{p}^{(t)})}}$$

$$p_j^{(t+1)} \doteq \frac{1}{1 + \frac{1-p_j^{(t)}}{p_j^{(t)}} e^{-\eta z_j(\boldsymbol{p}^{(t+0.5)})}}$$

This kernel function has three desirable properties. First, the price updates associated with the logistic loss require no projection and as such the logistic **extratâtonnement** process is a highly natural price adjustment process. Second, the logistic divergence is not homogeneous, and as such Bregman continuity w.r.t. the logistic loss does not imply that the excess demand is constant. Third, the value of the logistic divergence heads to infinity as one of the prices heads to 0, i.e., for all $\boldsymbol{p} \in [0,1]^m$, $\lim_{\boldsymbol{q}\to\boldsymbol{0}} \text{div}_{h_{\text{LL}}}(\boldsymbol{p},\boldsymbol{q}) \to \infty$, which is highly desirable since for many Walrasian economies the excess demand when the price of any good heads to 0 is strictly positive.

Remark 5.4.4 [Contributions, and Connection to Impossibility Results]**.**

To the best of our knowledge, this is the most general convergence guarantee for (natural) price adjustment processes in Walrasian economies. As we will show in Chapter 10, this result also implies convergence of price adjustment processes to a Walrasian equilibrium in the canonical class of Walrasian economies known as Arrow-Debreu economies which have a Bregman-continuous excess demand function. This, in turn makes the mirror *extratâtonnement* process the first price adjustment process with global polynomial-time convergence to an Walrasian equilibrium of Arrow-Debreu economies beyond those that satisfy the weak gross substitutes condition.

The above result might at first seem in contradiction with the impossibility result of Papadimitriou and Yannakakis (2010), which states that for any price adjustment process $\boldsymbol{\pi}$, there exists a balanced economy $(m, \boldsymbol{z})$ which is λ-Lipschitz-continuous on Δ_m and $\boldsymbol{\pi}$ fails to converge to a ε-Walrasian equilibrium in $\mathrm{poly}(\frac{1}{\varepsilon})$ evaluations of the excess demand function. However, our result is not in contradiction with this result, since in contrast to Papadimitriou and Yannakakis (2010), we assume that the Walrasian economy is Lipschitz-continuous on $[0, 1]^m$ rather than Δ_m. As such, as our continuity assumption is stronger, our computational result does not contradict the impossibility result of Papadimitriou and Yannakakis (2010). Further, our result overcomes this impossibility result as it demonstrates that the computational impossibility results for Walrasian equilibria are due to "edge case" Walrasian economies for which the excess demand is not sufficiently continuous and that in general the polynomial-time computation of a Walrasian equilibrium via price adjustment processes is possible.

Further, this result is also not in contradiction with PPAD-hardness result for the computation of Arrow-Debreu equilibrium prices in Leontief Arrow-Debreu economies (Codenotti et al., 2006; Deng and Du, 2008) and additively separable, piecewise linear and concave Arrow-Debreu economies (Chen et al., 2009), since without further assumptions for such economies the excess demand $\boldsymbol{z}$ can only be shown to be Lipschitz-continuous on Δ_m, and not on $[0, 1]^m$.

Beyond the results obtained in this section to obtain a stronger convergence result, we have to restrict our attention to competitive economies which satisfy WARP.

5.4.2 Computation of Walrasian Equilibrium in Variationally Stable Competitive Economies

In light of Lemma 5.4.2, a sensible question to investigate whether the mirror **extratâtonnement** process can be guaranteed to converge when the price space is chosen to be $\mathcal{P} \doteq \Delta_m$ thus guaranteeing Bregman-continuity of the excess demand since we have $\max_{\boldsymbol{p} \in \Delta} \frac{1}{\max_{k \in [m]} p_k} = \frac{1}{m}$. However, the set of Walrasian equilibria of any balanced economy $(m, \mathcal{Z})$ is not necessarily a subset of the strong solutions of the VI

$(\Delta_m, \mathcal{Z})$. Nevertheless, if the economy $(m, \mathcal{Z})$ is assumed to be competitive, then by Theorem 5.2.3 the set of strong solutions of the VI $(\Delta_m, \mathcal{Z})$ is a subset of the set of Walrasian equilibria of $(m, \mathcal{Z})$.

Unfortunately, as the price space $\mathcal{P} = \Delta_m$ does not include the zero vector $\mathbf{0}_m$ which ensures that balanced economies are variationally stable, the restriction of the price space to Δ_m effectively "destabilizes" the economy and makes computation of a Walrasian equilibrium intractable. As a result, to overcome this challenge we have to restrict the class of competitive economies to the class of competitive economies which are variationally stable on Δ_m.

Remark 5.4.5 [Interpretting Minty's condition]**.**
For balanced economies, by weak Walras' law a sufficient condition for the economy to be variationally stable on Δ_m is the existence of $\boldsymbol{p}^* \in \Delta_m$ s.t. for all prices $\boldsymbol{p} \in \Delta_m$, $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p})$:

$$\langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p}^* \rangle \geq 0$$

Now, suppose that there exists a commodity $j \in [m]$ which is (weakly) demanded in excess for all $\boldsymbol{p} \in \Delta_m$, i.e., $z_j(\boldsymbol{p}) \geq 0$. Then, setting $\boldsymbol{p}^* = \boldsymbol{j}_j$, we have $\langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p}^* \rangle = \langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{j}_j \rangle = z_j(\boldsymbol{p}) \geq 0$. Hence, if there is a good which is never supplied in excess the economy is variationally stable.

Alternatively, a balanced economy is variationally stable on Δ_m whenever there exists two commodities $j, k \in [m]$, whose excess demands are negatively proportional for all prices, i.e., $\exists \alpha > 0$, s.t. $z_j(\boldsymbol{p}) \geq -\alpha z_k(\boldsymbol{p})$, Then, setting $\boldsymbol{p}^* = \frac{1}{1+\alpha}\boldsymbol{j}_j + \frac{\alpha}{(1+\alpha)}\boldsymbol{j}_k$, we have $\langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p}^* \rangle = \frac{1}{1+\alpha} z_j(\boldsymbol{p}) + \frac{\alpha}{(1+\alpha)} z_k(\boldsymbol{p}) \geq \frac{-\alpha}{1+\alpha} z_k(\boldsymbol{p}) + \frac{\alpha}{(1+\alpha)} z_k(\boldsymbol{p}) = 0$. In light of this observation, the variational stability assumption on Δ_m can be seen as a rather mild assumption, as commodities whose excess demands are negatively correlated are abundant in the real world. For instance, airplane tickets and airplanes, whenever the excess demand for airline tickets is positive, this must mean that there are not enough airplanes, that is the excess demand for planes is negative.

We now discuss some important classes of Walrasian economies which are variationally stable on Δ_m. The most basic class of Walrasian economies which are variationally stable on Δ_m are those which satisfy the law of supply and demand. Intuitively, these Walrasian economies are those for which the excess demand is downward sloping.

Definition 5.4.8 [Law of supply and demand economies]**.**
Given a Walrasian economy $(m, \mathcal{Z})$, an excess demand correspondence is said to satisfy the **law of supply and demand** iff

$$\langle \boldsymbol{z}(\boldsymbol{q}) - \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{q} - \boldsymbol{p} \rangle \leq 0 \qquad\qquad \text{for all } \boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p}), \boldsymbol{z}(\boldsymbol{q}) \in \mathcal{Z}(\boldsymbol{q}) \tag{5.13}$$

If the excess demand correspondence of a Walrasian economy satisfies the law of supply and demand, we refer to the economy colloquially as a **law of supply and demand economy**.

We note that the excess demand of a Walrasian economies satisfies the law of supply and demand iff $-\mathcal{Z}$ is monotone. This implies that $-\mathcal{Z}$ is quasimonotone, and hence for any non-empty and compact price space $\mathcal{P} \subseteq \mathbb{R}^m_+$ the VI $(\mathcal{P}, -\mathcal{Z})$ satisfies the Minty condition (see Lemma 3.1 of He (2017)), meaning that any Walrasian economy which satisfies the law of supply and demand is variationally stable on $\mathcal{P}$.

Another important class of Walrasian economies which are variationally stable on Δ_m is the class of Walrasian economies which satisfy the weak gross substitutes condition. Intuitively, these are Walrasian economies for which the excess demand for a given good only increases when the price of some other good increases. While we omit the proof as it is involved, we note that any continuous balanced weak gross substitutes Walrasian economy $(m, \mathcal{Z})$ which satisfies Walras' law, i.e., (for all $\boldsymbol{p} \in \mathbb{R}^m_+, \boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p}), \boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p})$) is a subset of the class of variationally stable economies on $\mathcal{P} \subseteq \mathbb{R}^m_+$ for any non-empty and compact price space $\mathcal{P}$ (see, for instance lemma 5 of Arrow et al. (1959)).

Definition 5.4.9 [Weak Gross Substitutes economies]**.**
Given a Walrasian economy $(m, \mathcal{Z})$, an excess demand correspondence is said to satisfy the **weak gross substitutes condition (GS)** iff for all $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^m_+$ s.t. for some $k \in [m]$, $q_k > p_k$ and for all $j \neq k, q_j = p_j$, we have:

$$z_j(\boldsymbol{q}) \geq z_j(\boldsymbol{p}) \qquad\qquad \text{for all } \boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p}), \boldsymbol{z}(\boldsymbol{q}) \in \mathcal{Z}(\boldsymbol{q}) \tag{5.14}$$

If the above inequality holds strictly, then the excess demand is said to satisfy the gross substitutes condition (GS). Further, if the excess demand correspondence of a Walrasian economy satisfies WGS (resp. GS), we refer to the economy colloquially as a **WGS (resp. GS) economy**.

Going further, we can show that any Walrasian economy which satisfies the well-known weak axiom of revealed preferences Afriat (1967); Arrow and Hurwicz (1958), is variationally stable on Δ_m (and more generally on any non-empty and compact price space $\mathcal{P} \subseteq \mathbb{R}^m$). To this end, let us first define the weak axiom of revealed preferences for balanced economies.

Definition 5.4.10 [WARP excess demand]**.**

Given a Walrasian economy $(m, \mathcal{Z})$, an excess demand correspondence is said to satisfy the **weak axiom of revealed preferences** (**WARP**) iff for all $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p}), \boldsymbol{z}(\boldsymbol{q}) \in \mathcal{Z}(\boldsymbol{q})$:

$$\langle \boldsymbol{z}(\boldsymbol{q}), \boldsymbol{p} \rangle \leq \langle \boldsymbol{z}(\boldsymbol{q}), \boldsymbol{q} \rangle \text{ and } \boldsymbol{z}(\boldsymbol{p}) \neq \boldsymbol{z}(\boldsymbol{q}) \implies \langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{q} \rangle > \langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p} \rangle$$

If the excess demand correspondence of a Walrasian economy satisfies WARP, we refer to the economy colloquially as a **WARP economy**.

Remark 5.4.6.

This definition of (WARP) is adapted to arbitrary Walrasian economies and as such is a generalization of the usual definition for economies which satisfy Walras' law (i.e., for all $\boldsymbol{p} \in \mathbb{R}^m_+$, $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p})$), which requires that $\mathcal{Z}$ is singleton-valued, and $\langle \boldsymbol{z}(\boldsymbol{q}), \boldsymbol{p} \rangle \leq 0$ and $\boldsymbol{z}(\boldsymbol{p}) \neq \boldsymbol{z}(\boldsymbol{q}) \implies \langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{q} \rangle > 0$ (i.e., for all $\boldsymbol{p} \in \mathbb{R}^m_+$, $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p})$).

As we show next, WARP implies that $-\mathcal{Z}$ is pseudomonotone in balanced economies.[6]

Lemma 5.4.3 [WARP $\implies$ pseudomonotone]**.**

If the excess demand correspondence $\mathcal{Z}$ of a Walrasian economy $(m, \mathcal{Z})$ satisfies WARP, then $-\mathcal{Z}$ is pseudomonotone.

Proof

Suppose that $\mathcal{Z}$ satisfies WARP, and that $\langle -\boldsymbol{z}(\boldsymbol{q}), \boldsymbol{q} - \boldsymbol{p} \rangle = \langle \boldsymbol{z}(\boldsymbol{q}), \boldsymbol{p} - \boldsymbol{q} \rangle \leq 0$, then we have $\langle -\boldsymbol{z}(\boldsymbol{p}), \boldsymbol{q} - \boldsymbol{p} \rangle = \langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p} - \boldsymbol{q} \rangle$

If $\boldsymbol{z}(\boldsymbol{p}) \neq \boldsymbol{z}(\boldsymbol{q})$, then, by WARP, we have $\langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p} - \boldsymbol{q} \rangle < 0$.

Otherwise, if $\boldsymbol{z}(\boldsymbol{p}) = \boldsymbol{z}(\boldsymbol{q})$, then we have:

$$\langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p} - \boldsymbol{q} \rangle = \langle \boldsymbol{z}(\boldsymbol{q}), \boldsymbol{p} - \boldsymbol{q} \rangle \leq 0$$

[6] To be more precise, we note that an excess demand function $\mathcal{Z}$ satisfies WARP iff $-\mathcal{Z}$ is strictly pseudomonotone. However, as this result will not be used we present the more general result.

That is, if $\mathcal{Z}$ satisfies WARP, we have:

$$\langle -\boldsymbol{z}(\boldsymbol{q}), \boldsymbol{q} - \boldsymbol{p} \rangle \leq 0 \implies \langle -\boldsymbol{z}(\boldsymbol{p}), \boldsymbol{q} - \boldsymbol{p} \rangle \leq 0$$

Hence, $-\mathcal{Z}$ is pseudomonotone.

An important consequence of Lemma 5.4.3 is that since $-\mathcal{Z}$ is pseudomonotone, for any non-empty and compact price space $\mathcal{P} \subseteq \mathbb{R}^m_+$ the VI $(\mathcal{P}, -\mathcal{Z})$ satisfies the Minty condition (see Lemma 3.1 of He (2017)). As such, we have the following corollary of Lemma 5.4.3.

Corollary 5.4.3 [WARP $\implies$ Variationally Stable]**.**
Any Walrasian economy which satisfies WARP is variationally stable on any non-empty and compact price space $\mathcal{P} \subseteq \mathbb{R}^m_+$.

To use Lemma 5.4.2 we have to ensure that the excess demand of the economy is bounded, which as we will show in Chapter 10, is a mild assumption satisfied in all Arrow-Debreu economies, thus necessitating the following definition.

Definition 5.4.11 [Bounded economies]**.**
Given $\overline{z} \geq 0$, a $\overline{z}$**-bounded economy** $(m, \boldsymbol{d}, \boldsymbol{s})$ is a Walrasian economy $(m, \mathcal{Z})$ which consists of an **aggregate demand function** $\boldsymbol{d} : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ and an **aggregate supply function** $\boldsymbol{s} : \mathbb{R}^m_+ \to \mathbb{R}^m_+$ s.t. we have $\mathcal{Z}(\boldsymbol{p}) \doteq \{\boldsymbol{d}(\boldsymbol{p})\} - \{\boldsymbol{s}(\boldsymbol{p})\}$ and the following bounds hold:

$$\|\boldsymbol{d}\|_\infty \leq \overline{z} \qquad\qquad \|\boldsymbol{s}\|_\infty \leq \overline{z}$$

With definitions in place, we can now apply Lemma 5.4.2 to derive the polynomial-time convergence of the mirror extrâtonnement process in conjunction with Theorem 5.4.1.

Theorem 5.4.2 [Mirror Extratâtonnement Convergence in Δ_m]**.**
Let $(m, \boldsymbol{d}, \boldsymbol{e})$ be a $\overline{\epsilon}$-elastic and $\overline{z}$-bounded balanced economy which is variationally stable on Δ_m. Consider the mirror *extrâtonnement* process run on $(m, \boldsymbol{z})$, with a 1-strongly-convex and κ-Lipschitz-smooth kernel function h, any time horizon $t \mathrel{\dot\in} \mathbb{N}$, any step size $\eta \in (0, \frac{1}{2\sqrt{2}m\epsilon\overline{z}}]$, a price space $\mathcal{P} \doteq \Delta_m$, and any initial price vector $\boldsymbol{p}^{(0)} \in \Delta_m$, and let $\{\boldsymbol{p}^{(t)}, \boldsymbol{p}^{(t+0.5)}\}_t$ be the sequence of prices generated. The following convergence

bound holds:

$$\min_{k=0,\dots,\tau} \max_{\boldsymbol{p}\in\Delta} \langle \boldsymbol{z}(\boldsymbol{p}^{(k+0.5)}), \boldsymbol{p} - \boldsymbol{p}^{(k+0.5)} \rangle \leq \frac{2\sqrt{2}(1+\kappa)}{\eta} \frac{\sqrt{\max_{\boldsymbol{p}\in\Delta} \mathrm{div}_h(\boldsymbol{p}, \boldsymbol{p}^{(0)})}}{\sqrt{\tau}}$$

Further, we have that $\lim_{t\to\infty} \boldsymbol{p}^{(t+0.5)} = \lim_{t\to\infty} \boldsymbol{p}^{(t)} = \boldsymbol{p}^*$ is a Walrasian equilibrium.

Proof of Theorem 5.4.2

Since $(m, \boldsymbol{z})$ is variationally stable on Δ_m, the VI $(\Delta_m, -\mathcal{Z})$ satisfies the Minty condition. In addition, since by the assumption of the theorem the economy is $\bar{\epsilon}$-elastic and $\bar{z}$-bounded, by Lemma 5.4.2, $\boldsymbol{z}$ is $(2m\epsilon\bar{z})$-Bregman-continuous on Δ_m. That is, we have

$$\begin{aligned}
\frac{1}{2}\|\boldsymbol{z}(\boldsymbol{q}) - \boldsymbol{z}(\boldsymbol{p})\|^2 &\leq \left(\frac{\epsilon\left(\|\boldsymbol{d}(\boldsymbol{p})\| + \|\boldsymbol{s}(\boldsymbol{p})\|\right)}{\|\boldsymbol{p}\|_\infty}\right)^2 \mathrm{div}_h(\boldsymbol{q}, \boldsymbol{p}) \\
&\leq \max_{\boldsymbol{p}\in\Delta_m} \left(\frac{\epsilon\left(\|\boldsymbol{d}(\boldsymbol{p})\| + \|\boldsymbol{s}(\boldsymbol{p})\|\right)}{\|\boldsymbol{p}\|_\infty}\right)^2 \mathrm{div}_h(\boldsymbol{q}, \boldsymbol{p}) \\
&\leq \left(\frac{\epsilon\left(\|\boldsymbol{d}\|_\infty + \|\boldsymbol{s}\|_\infty\right)}{\min_{\boldsymbol{p}\in\Delta_m} \|\boldsymbol{p}\|_\infty}\right)^2 \mathrm{div}_h(\boldsymbol{q}, \boldsymbol{p}) \\
&\leq \left(\frac{2\epsilon\bar{z}}{\frac{1}{m}}\right)^2 \mathrm{div}_h(\boldsymbol{q}, \boldsymbol{p}) \\
&\leq (2m\epsilon\bar{z})^2 \,\mathrm{div}_h(\boldsymbol{q}, \boldsymbol{p}) \ .
\end{aligned}$$

That is,

Suppose that under the assumptions of the theorem the mirror generates the sequence of prices $\{\boldsymbol{p}^{(t)}, \boldsymbol{p}^{(t+0.5)}\}_t$. Let $\boldsymbol{p}^{(\tau)}_{\text{best}} \in \arg\min_{\boldsymbol{x}^{(k+0.5)}:k=0,\dots,\tau} \mathrm{div}_h(\boldsymbol{p}^{(k+0.5)}, \boldsymbol{p}^{(k)})$. As the mirror *extratâtonnement* process is simply the mirror extragradient method run on the VI $(\Delta_m, -\mathcal{Z})$, and the assumptions of Theorem 4.3.1 are satisfied and hence we have the following bound:

$$\min_{k=0,\dots,\tau} \max_{\boldsymbol{p}\in\Delta} \langle -\boldsymbol{z}(\boldsymbol{p}^{(k+0.5)}), \boldsymbol{p}^{(k+0.5)} - \boldsymbol{p} \rangle \leq \frac{2(1+\kappa)\mathrm{diam}(\Delta_m)}{\eta} \frac{\sqrt{\max_{\boldsymbol{p}\in\Delta} \mathrm{div}_h(\boldsymbol{p}, \boldsymbol{p}^{(0)})}}{\sqrt{\tau}}$$

$$\min_{k=0,\dots,\tau} \max_{\boldsymbol{p}\in\Delta} \langle \boldsymbol{z}(\boldsymbol{p}^{(k+0.5)}), \boldsymbol{p} - \boldsymbol{p}^{(k+0.5)} \rangle \leq \frac{2\sqrt{2}(1+\kappa)}{\eta} \frac{\sqrt{\max_{\boldsymbol{p}\in\Delta} \mathrm{div}_h(\boldsymbol{p}, \boldsymbol{p}^{(0)})}}{\sqrt{\tau}}$$

Further, $\lim_t \boldsymbol{p}^{(t)} = \lim_{t\to\infty} \boldsymbol{p}^{t+0.5} = \boldsymbol{p}^*$ is a Walrasian equilibrium.

With this theorem in hand, we make the following remarks before turning our attention to second order price adjustment processes.

Remark 5.4.7 [Contribution]**.**

While Arrow and Hurwicz (1958) had in a seminal paper shown that tâtonnement price adjustment process converge in continuous time to a Walrasian equilibrium in Walrasian economies which satisfy WARP, to the best of our knowledge such a result did not exist in discrete time. As such, our result is the first polynomial-time computation result for ε-Walrasian equilibrium, and first convergence result for a price adjustment in the class of Arrow-Debreu economies which satisfy WARP.

Remark 5.4.8 [Boundedness of excess demand]**.**

The assumption that there exists $\overline{z} \geq 0$ s.t. for all $t \in [\tau]$, $\|\boldsymbol{z}(\boldsymbol{p}^{(t)})\| \leq \overline{z}$ is a common place assumption in the analysis of discrete time price adjustment processes (see, for instance, Cheung et al. (2013) or Chapter 6), and is often guaranteed by doing a more fine grained analysis of the Walrasian economy at hand. That said, it is indeed possible to restrict the excess demand to bounded by the total amount of the commodity that can be ever supplied without modifying the Walrasian equilibria of the economy. This is indeed the approach that Arrow and Debreu (1954) take in Section 3 of their paper for proving their seminal Walrasian equilibrium existence result. This restriction is also realistic from an economic perspective since it is not possible for the economy to consume more of a commodity that there can exist, and resources in the real-world are indeed scarce. Indeed, otherwise there would be no use for the economic sciences: the science of resource allocation under scarcity. We present the result in this format to maintain generality of the results for future work.

Remark 5.4.9 [Local convergence of mirror *extratâtonnement*]**.**

The local convergence behavior of mirror *extratâtonnement* can similarly be inferred by this result by instead applying Theorem 4.3.2, and replacing the assumption that the Arrow-Debreu satisfies the Minty condition with the assumption that the initial price iterate starts close enough to a local Minty solution.

Mirror Extratâtonnement in Scarf Economies

One of the earliest negative and most discouraging results in the literature on price-adjustment processes is an example of a Walrasian economy provided by Herbert Scarf in which continuous-time tâtonnement is known to cycle around the Walrasian equilibrium of the economy, while discrete-time variants are known to spiral away from the equilibrium for any initial non-equilibrium price vector.

Definition 5.4.12.

A **Scarf economy** $\boldsymbol{z}^{\text{scarf}}$ is a Walrasian economy $(3, \boldsymbol{z}^{\text{scarf}})$ with 3 goods for which the excess demand is singleton-valued and given by the function:

$$\boldsymbol{z}^{\text{scarf}}(\boldsymbol{p}) \doteq \begin{pmatrix} \frac{p_1}{p_1+p_2} + \frac{p_3}{p_1+p_3} - 1 \\ \frac{p_1}{p_1+p_2} + \frac{p_2}{p_2+p_3} - 1 \\ \frac{p_2}{p_2+p_3} + \frac{p_3}{p_1+p_3} - 1 \end{pmatrix}$$

The following lemma summarizes properties of the Scarf economy.

Lemma 5.4.4 [Properties of Scarf Economy]**.**

The Scarf economy is a balanced economy which satisfies Walras' law, i.e., for all $\boldsymbol{p} \in \mathbb{R}_+^m$, $\boldsymbol{p} \cdot \boldsymbol{z}^{\text{scarf}}(\boldsymbol{p}) = 0$. Further, the set of Walrasian equilibrium of the Scarf economy $\boldsymbol{z}^{\text{scarf}}$ is given by $\mathcal{WE}(\boldsymbol{z}^{\text{scarf}}) \doteq \{\lambda \mathbf{1}_3 \mid \lambda > 0\}$.

Proof of Lemma 5.4.4

First, notice that the scarf economy is homogeneous of degree 0. That is, for all $\lambda \geq 0$, we have:

$$\boldsymbol{z}^{\text{scarf}}(\lambda\boldsymbol{p}) \doteq \begin{pmatrix} \frac{\lambda p_1}{\lambda p_1+\lambda p_2} + \frac{\lambda p_3}{\lambda p_1+\lambda p_3} - 1 \\ \frac{\lambda p_1}{\lambda p_1+\lambda p_2} + \frac{\lambda p_2}{\lambda p_2+\lambda p_3} - 1 \\ \frac{\lambda p_2}{\lambda p_2+\lambda p_3} + \frac{\lambda p_3}{\lambda p_1+\lambda p_3} - 1 \end{pmatrix} = \begin{pmatrix} \frac{p_1}{p_1+p_2} + \frac{p_3}{p_1+p_3} - 1 \\ \frac{p_1}{p_1+p_2} + \frac{p_2}{p_2+p_3} - 1 \\ \frac{p_2}{p_2+p_3} + \frac{p_3}{p_1+p_3} - 1 \end{pmatrix} = \boldsymbol{z}^{\text{scarf}}(\boldsymbol{p})$$

Second, for all $\boldsymbol{p} \in \mathbb{R}^m$, notice we have:

$$\begin{aligned} \boldsymbol{p} \cdot \boldsymbol{z}^{\text{scarf}}(\boldsymbol{p}) &= \frac{p_1^2}{p_1+p_2} + \frac{p_1 p_3}{p_1+p_3} - p_1 + \frac{p_1 p_2}{p_1+p_2} + \frac{p_2^2}{p_2+p_3} - p_2 + \frac{p_2 p_3}{p_2+p_3} + \frac{p_3^2}{p_1+p_3} - p_3 \\ &= \frac{p_1^2 + p_1 p_2}{p_1+p_2} + \frac{p_2^2 + p_2 p_3}{p_2+p_3} + \frac{p_3^2 + p_1 p_3}{p_1+p_3} - p_1 - p_2 - p_3 \\ &= \frac{p_1(p_1+p_2)}{p_1+p_2} + \frac{p_2(p_2+p_3)}{p_2+p_3} + \frac{p_3(p_3+p_1)}{p_1+p_3} - p_1 - p_2 - p_3 \\ &= 0 \end{aligned}$$

Finally, observe that for $\boldsymbol{p}^* = \mathbf{1}_m$, we have $\boldsymbol{z}^{\text{scarf}}(\boldsymbol{p}^*) = \mathbf{0}_m$, and, we have $\boldsymbol{p}^* \cdot \boldsymbol{z}^{\text{scarf}}(\boldsymbol{p}^*)$. Notice that this equilibrium is unique up to positive scaling since if the price of any commodity is changed from $\boldsymbol{p}^*$, then the excess demand for another commodity is guaranteed to decrease while the excess demand of some other commodity increases.

Our next result shows that the Scarf economy is variationally stable and Lipschitz-continuous for any suitably chosen price space.

Lemma 5.4.5 [Variational stability and Bregman-continuity of the Scarf Economy]**.**
Any Scarf economy $\boldsymbol{z}^{\text{scarf}}$ is variationally stable on Δ_m. Further, for any $\underline{p} \in (0, 1/3)$ and any 1-strongly-convex kernel function $h : \mathbb{R}^3_+ \to \mathbb{R}$, the Scarf economy $\boldsymbol{z}^{\text{scarf}}$ is variationally stable and $(\frac{3}{\underline{p}^2}, h)$-Bregman-continuous on $[\underline{p}, 1]^3$.

Proof of Lemma 5.4.5

Part 1: Variational stability on Δ_m. We claim that for $\boldsymbol{p}^* = (1/3, 1/3, 1/3)$, the Scarf economy is variationally stable on Δ_m, i.e., we have for all prices $\boldsymbol{p} \in \Delta_3$ and all $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p})$, we have $\left\langle \boldsymbol{z}^{\text{scarf}}(\boldsymbol{p}), \boldsymbol{p}^* - \boldsymbol{p} \right\rangle \geq 0$.

First, notice that expanding the expression $\left\langle \boldsymbol{z}^{\text{scarf}}(\boldsymbol{p}), \boldsymbol{p}^* - \boldsymbol{p} \right\rangle$ we have for all $\boldsymbol{p} \in \Delta_m$:

$$\begin{aligned}
\left\langle \boldsymbol{z}^{\text{scarf}}(\boldsymbol{p}), \boldsymbol{p}^* - \boldsymbol{p} \right\rangle &= \left\langle \boldsymbol{z}^{\text{scarf}}(\boldsymbol{p}), \boldsymbol{p}^* \right\rangle - \underbrace{\left\langle \boldsymbol{z}^{\text{scarf}}(\boldsymbol{p}), \boldsymbol{p} \right\rangle}_{=0} \\
&= \left\langle \boldsymbol{z}^{\text{scarf}}(\boldsymbol{p}), \boldsymbol{p}^* \right\rangle \\
&= 2\frac{p_1}{p_1 + p_2} + 2\frac{p_2}{p_2 + p_3} + 2\frac{p_3}{p_1 + p_3} - \underbrace{1/3 - 1/3 - 1/3}_{=-1} \\
&= 2\frac{p_1}{p_1 + p_2} + 2\frac{p_2}{p_2 + p_3} + 2\frac{p_3}{p_1 + p_3} - 1
\end{aligned}$$

We proceed by proof by cases.

Case 1 $p_1 \geq p_2$**.**

$$\begin{aligned}
\left\langle \boldsymbol{z}^{\text{scarf}}(\boldsymbol{p}), \boldsymbol{p}^* - \boldsymbol{p} \right\rangle &= 2\frac{p_1}{p_1 + p_2} + \underbrace{2\frac{p_2}{p_2 + p_3}}_{\geq 0} + \underbrace{2\frac{p_3}{p_1 + p_3}}_{\geq 0} - 1 \\
&\geq 2\frac{p_1}{p_1 + \underbrace{p_2}_{\leq p_1}} - 1 \\
&\geq 2\frac{p_1}{p_1 + p_1} - 1 \\
&= 1 - 1 \\
&= 0
\end{aligned}$$

Case 2 $p_1 < p_2$**.**

$$\begin{aligned}\left\langle \boldsymbol{z}^{\text{scarf}}(\boldsymbol{p}), \boldsymbol{p}^* - \boldsymbol{p} \right\rangle &= \underbrace{2\frac{p_1}{p_1 + p_2}}_{\geq 0} + 2\frac{p_2}{p_2 + p_3} + 2\frac{p_3}{p_1 + p_3} - 1 \\ &= 2\frac{p_2}{p_2 + p_3} + 2\frac{p_3}{\underbrace{p_1}_{<p_2} + p_3} - 1 \\ &= 2\frac{p_2}{p_2 + p_3} + 2\frac{p_3}{p_2 + p_3} - 1 \\ &= 2\frac{p_2 + p_3}{p_2 + p_3} - 1 \\ &= 2 - 1 \\ &\geq 0\end{aligned}$$

Hence, we must have: $\left\langle \boldsymbol{z}^{\text{scarf}}(\boldsymbol{p}), \boldsymbol{p}^* - \boldsymbol{p} \right\rangle \geq 0$, and the Scarf economy is variationally stable on Δ_m.

Part 2: Variational stability and Bregman-continuity on $[\underline{p}, 1]^3$**.** First, for variational stability on $[\underline{p}, 1]$, observe that the proof provided in part 1 applies directly by replacing Δ_m by $[\underline{p}, 1]$.

Second, notice that the excess demand is differentiable with its Jacobian matrix given by:

$$\nabla \boldsymbol{z}(\boldsymbol{p}) = \begin{bmatrix} -\frac{p_1}{(p_1+p_2)^2} - \frac{p_3}{(p_1+p_3)^2} & -\frac{p_1}{(p_1+p_2)^2} & -\frac{p_3}{(p_1+p_3)^2} \\ -\frac{p_1}{(p_1+p_2)^2} & -\frac{p_1}{(p_1+p_2)^2} - \frac{p_3}{(p_2+p_3)^2} & -\frac{p_2}{(p_2+p_3)^2} \\ -\frac{p_3}{(p_1+p_3)^2} & -\frac{p_3}{(p_2+p_3)^2} & -\frac{p_2}{(p_2+p_3)^2} - \frac{p_3}{(p_1+p_3)^2} \end{bmatrix}$$

Thus, the Jacobian consists of entries of the form of $f(x, y) \doteq \frac{x}{(x+y)^2}$. For $x, y \in [\underline{p}, 1]$, we then have $|f(x, y)| \leq \frac{1}{4\underline{p}^2}$. This means that the absolute value of the off diagonal entries of $\nabla \boldsymbol{z}(\boldsymbol{p})$ are bounded by $\frac{1}{4\underline{p}^2}$, while the diagonal entries are bounded by $\frac{1}{2\underline{p}^2}$. Hence, for all $\boldsymbol{p} \in [\underline{p}, 1]^3$, we have $\|\nabla \boldsymbol{z}(\boldsymbol{p})\|_1 \leq \frac{3}{2\underline{p}^2} + \frac{6}{4\underline{p}^2} = \frac{3}{\underline{p}^2}$. Then, by the mean value theorem, $\boldsymbol{z}^{\text{scarf}}$ is $\frac{3}{\underline{p}^2}$-Lipschitz-continuous on $[\underline{p}, 1]^3$, i.e., for all $\boldsymbol{p}, \boldsymbol{q} \in [\underline{p}, 1]^3$, $\|\boldsymbol{z}(\boldsymbol{p}) - \boldsymbol{z}(\boldsymbol{q})\| \leq \frac{3}{\underline{p}^2} \|\boldsymbol{q} - \boldsymbol{p}\|$. Now, by the assumptions of the theorem, since h is 1-strongly-convex, we have for all $\boldsymbol{p}, \boldsymbol{q} \in \mathbb{R}^3_+$, $\frac{1}{2}\|\boldsymbol{p} - \boldsymbol{q}\|^2 \leq \text{div}_h(\boldsymbol{p}, \boldsymbol{q})$. Hence, this implies that for all for all $\boldsymbol{p}, \boldsymbol{q} \in [\underline{p}, 1]^3$,

$$\begin{aligned} {}^1\!/_2\|\boldsymbol{z}(\boldsymbol{p}) - \boldsymbol{z}(\boldsymbol{q})\|^2 &\leq \frac{1}{2}\left(\frac{3}{\underline{p}^2}\right)^2 \|\boldsymbol{p} - \boldsymbol{q}\|^2 \\ &\leq \left(\frac{3}{\underline{p}^2}\right)^2 \operatorname{div}_h(\boldsymbol{p}, \boldsymbol{q}) \end{aligned}$$

With the above lemma in hand, we can then prove the convergence of mirror *extratâtonnement* process in the Scarf economy.

Corollary 5.4.4 [Convergence of Mirror *Extrâtonnement* in Scarf Economies]**.**
Let $\underline{p} \in (0, 1)$. Consider the mirror *extrâtonnement* process run on the Scarf economy $\boldsymbol{z}^{\text{scarf}}$, with a 1-strongly-convex and κ-Lipschitz-smooth kernel function h, any time horizon $t \doteq \mathbb{N}$, any step size $\eta \in (0, \frac{1}{\sqrt{2}\lambda}]$, a price space $\mathcal{P} \doteq [\underline{p}, 1]^3$, and any initial price vector $\boldsymbol{p}^{(0)} \in \mathcal{P}$, and let $\{\boldsymbol{p}^{(t)}, \boldsymbol{p}^{(t+0.5)}\}_t$ be the sequence of prices generated. Then, we have that $\lim_{t\to\infty} \boldsymbol{p}^{(t+0.5)} = \lim_{t\to\infty} \boldsymbol{p}^{(t)} = \boldsymbol{p}^*$ is a Walrasian equilibrium.

5.4.3 Experiments for Mirror *Extratâtonnement* Process

In this section, we first apply the *tâtonnement* and mirror *extratâtonnement* process with kernel function $h(\boldsymbol{p}) \doteq \|\boldsymbol{p}\|^2$, to solve the Scarf economy with the goal of illustrating the differing convergence behavior between the two price-adjustment processes. We then apply the mirror *extratâtonnement* process with kernel function $h(\boldsymbol{p}) \doteq \|\boldsymbol{p}\|^2$ to solve a number of Arrow-Debreu exchange economies (Arrow and Debreu, 1954) with the goal of demonstrating that our pathwise Bregman-continuity assumption holds, and that the mirror *extratâtonnement* process can efficiently solve very large Walrasian economies in practice.

Table 5.1: Summary of Setups for Arrow-Debreu Exchange Economy Experiments

Exp No.	Num. Comm.	Num. Linear Cons.	Num. Cobb -Doug. Cons.	Num. CES $\rho \in (0, 1)$ Cons.	Num. CES $\rho < 0$ Cons.	Num. Leont. Cons.
1	500	0	0	0	0	600
2	500	0	0	0	600	0
3	500	0	0	600	0	0
4	500	0	600	0	0	0
5	500	600	0	0	0	0
6	1000	200	200	200	200	200
7	1000	0	200	200	200	200

We record in the first 2 leftmost plots of Figure 5.1 the movement of prices in the Scarf economy for the *tâtonnement* and mirror *extratâtonnement* processes respectively. As is well-established, the sequence of prices generated by *tâtonnement*, despite starting very close to the equilibrium prices $(1/3, 1/3, 1/3)$ spiral away from the prices, converging to the $(0, 0, 1)$ price vector which is not a Walrasian equilibrium. In contrast, the prices generated by the mirror *extratâtonnement* process spiral inwards towards the equilibrium price despite starting far away from the equilibrium prices. An intuitive explanation of this behavior is as follows, the continuous-time variant of *tâtonnement* is known to cycle around the equilibrium prices (Scarf, 1960). Now, one way to interpret the discrete-time *tâtonnement* (resp. mirror *extratâtonnement*) process is as an explicit (resp. implicit) discretization (Butcher, 2008) of the continuous-time *tâtonnement* dynamics. A well-known fact is that explicit (resp. implicit) discretization methods are unstable (resp. stable) when the continuous-time dynamics cycle, thus explaining the observed behavior.

An Arrow-Debreu exchange economy $(n, m, \mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$ consists of $m \in \mathbb{N}$ commodities, $n \in \mathbb{N}$ consumers each $i \in [n]$ with a consumption space $\mathcal{X}_i$, an endowment of commodities $\boldsymbol{e}_i \in \mathbb{R}_+^m$, and a utility function $u_i : \mathcal{X}_i \to \mathbb{R}$. An Arrow-Debreu exchange economy $(n, m, \mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$ can be represented as a bounded continuous competitive economy $(m, \mathcal{Z})$ where the excess demand correspondence is given as: $\mathcal{Z}(\boldsymbol{p}) \doteq \sum_{i \in [n]} \operatorname*{arg\,max}_{\boldsymbol{x}_i \in \mathcal{X}_i : \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p}} u_i(\boldsymbol{x}_i) - \sum_{i \in [n]} \boldsymbol{e}_i$.[7]

We consider the following utility function classes to run our experiments: 1. linear: $u_i(\boldsymbol{x}_i) = \sum_{j \in [m]} v_{ij} x_{ij}$; 2. Cobb-Douglas: $u_i(\boldsymbol{x}_i) = \prod_{j \in [m]} x_{ij}^{v_{ij}}$; 3. Leontief: $u_i(\boldsymbol{x}_i) = \min_{j \in [m]} \left\{ x_{ij}/v_{ij} \right\}$; and 4. CES: $u_i(\boldsymbol{x}_i) = \sqrt[\rho_i]{\sum_{j \in [m]} v_{ij} x_{ij}^{\rho_i}}$ with each utility function parameterized by a vector of valuations $\boldsymbol{v}_i \in \mathbb{R}_+^n$, where each v_{ij} quantifies the value of commodity j to consumer i. We summarize the experiments we run in Table 5.1. The parameters of each economy are initialized randomly according to the uniform random distribution.[8] We record the results of our experiments in Figure 5.1, describing for what value of $\varepsilon \geq 0$, are the prices generated throughout the algorithm a ε-Walrasian equilibrium.

We observe that in all our experiments except in experiments 5 and 6—which include Linear consumers and are as such not covered by our theory as the excess demand for the economies is not singleton-valued—the mirror *extratâtonnement* process converges to a Walrasian equilibrium. In all experiments, we verify and confirm that pathwise Bregman-continuity holds, thus justifying our assumption. Finally, while our experiments obey our theory which suggests a best-iterate convergence to a ε-Walrasian equilibrium in $1/\varepsilon^2$ time-steps, we observe that a last-iterate convergence occurs only for experiments 5, corresponding to the case of Cobb-Douglas consumers, for which even *tâtonnement* is known to converge in last-iterates. This

[7] We refer the reader to Chapter 10 on additional background and definitions on Arrow-Debreu exchange economies.

[8] For reproducibility purposes, we include our code ready to run on `https://github.com/denizalp/extratatonnement`, and include all details of our experimental setup in Section 5.4.3.

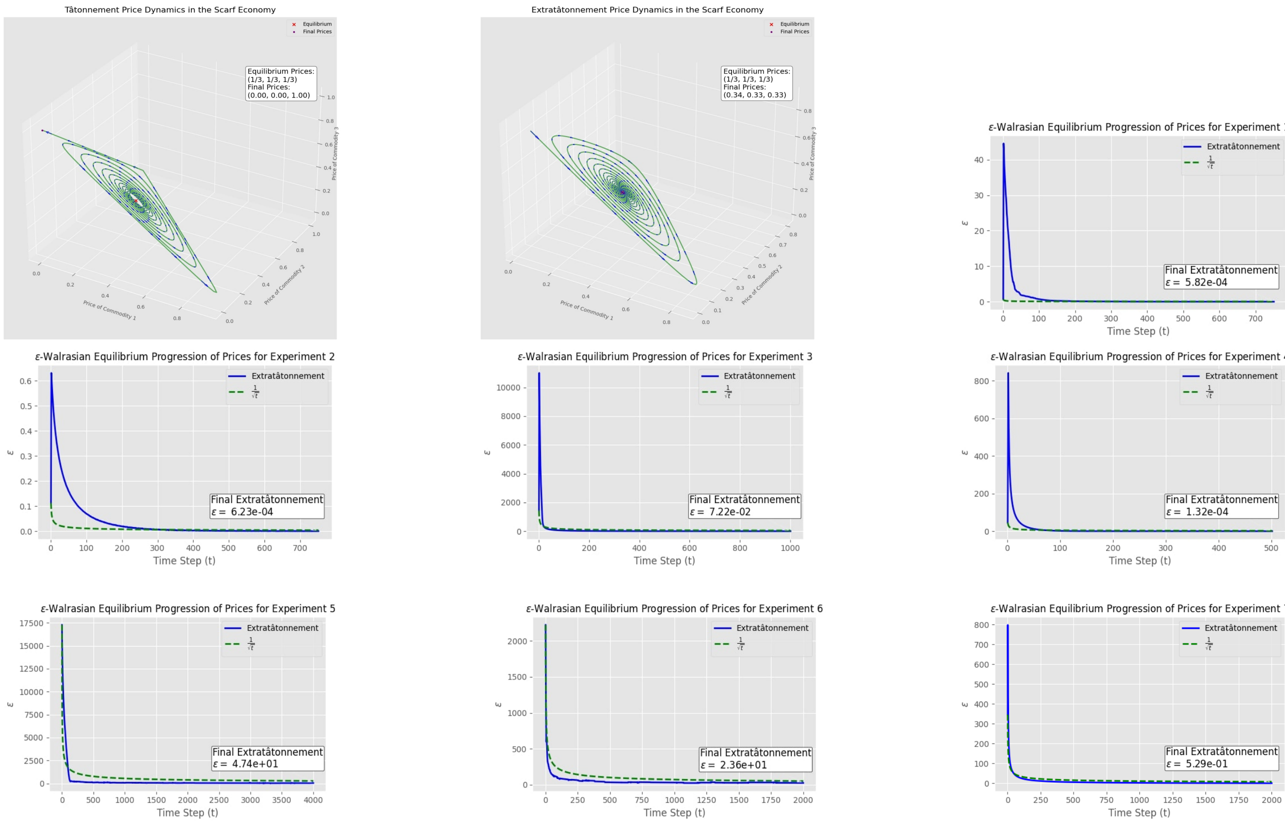


Figure 5.1: Phase Portraits of *Tâtonnement*, and *Extratâtonnement* for the Scarf Economy, and Results of Experiments 1-7.

suggests that achieve convergence in last iterates might not be possible with the mirror *extratâtonnement* process.

5.5 Merit Function Methods for Walrasian Equilibrium

5.5.1 Merit Function Minimization via Second-Order Price-Adjustment Process

Going beyond Minty Arrow-Debreu economies, we can also apply the merit function method we have introduced in Chapter 4 to solve general Walrasian economies with excess demand functions which are sufficiently smooth. Recall that by Theorem 5.2.1 the set of Walrasian equilibria of any Walrasian economy $(m, \boldsymbol{z})$ is equal to the set of strong solutions $\mathcal{SVI}(\mathbb{R}^m_+, -\boldsymbol{z})$ of the VI $(\mathbb{R}^m_+, -\boldsymbol{z})$. As such by using the merit function derived for VIs in Lemma 4.4.1, we define the following merit function function for Walrasian equilibrium:

$$\Xi_\alpha(\boldsymbol{p}) \doteq \max_{\boldsymbol{q} \in \mathbb{R}^m_+} \langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{q} - \boldsymbol{p} \rangle - \frac{\alpha}{2} \|\boldsymbol{q} - \boldsymbol{p}\|^2 \tag{5.15}$$

We have the following corollary of Lemma 4.4.1 which characterizes the above potential function.

Corollary 5.5.1 [Merit function for Walrasian equilibrium]**.**
Consider a Walrasian economy $(m, \boldsymbol{z})$. Then, for any $\alpha \geq 0$, the set of Walrasian equilibria of $(m, \mathcal{Z})$ is equal to $\arg\min_{\boldsymbol{p} \in \mathbb{R}^m_+} \Xi_\alpha(\boldsymbol{p})$. Further, suppose that $\alpha > 0$, then we have $\arg\max_{\boldsymbol{q} \in \mathbb{R}^m_+} \{\langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{q} - \boldsymbol{p} \rangle - \frac{\alpha}{2} \|\boldsymbol{q} - \boldsymbol{p}\|^2\} = \{\boldsymbol{q}^*(\boldsymbol{p})\}$ where:

$$\boldsymbol{q}^*(\boldsymbol{p}) = \arg\max_{\boldsymbol{q} \in \mathbb{R}^m_+} \langle \boldsymbol{z}(\boldsymbol{p}), \boldsymbol{p} - \boldsymbol{q} \rangle - \frac{\alpha}{2} \|\boldsymbol{q} - \boldsymbol{p}\|^2 = \Pi_{\mathbb{R}^m_+} \left[\boldsymbol{p} + \frac{1}{\alpha} \boldsymbol{z}(\boldsymbol{p}) \right] \ .$$

In addition, Ξ_α can be expressed as follows:

$$\Xi_\alpha(\boldsymbol{p}) = \max_{\boldsymbol{q} \in \mathbb{R}^m_+} \frac{\alpha}{2} \left[\left\| \frac{1}{\alpha} \boldsymbol{z}(\boldsymbol{p}) \right\|^2 - \left\| \boldsymbol{q} - \left(\boldsymbol{p} - \frac{1}{\alpha} \boldsymbol{f}(\boldsymbol{x}) \right) \right\|^2 \right] \ ,$$

with its gradient being given by:

$$\nabla \Xi_\alpha(\boldsymbol{p}) \doteq \boldsymbol{z}(\boldsymbol{p}) - (\nabla \boldsymbol{z}(\boldsymbol{p}) + \alpha \mathbb{I}) (\boldsymbol{q}^*(\boldsymbol{p}) - \boldsymbol{p}) \ .$$

5.5.2 Mirror Potential Algorithm for Walrasian Economies

With the above lemma in hand, we can minimize Ξ_α using the mirror potential algorithm (Algorithm 5). Note that as the mirror potential algorithm, the algorithm that arises is a second-order price adjustment process. The following corollary is obtained by applying Theorem 4.4.1 to the regularized primal gap as defined in Equation (5.15).

Theorem 5.5.1 [Mirror potential algorithm for Walrasian equilibrium]**.**

Consider a Walrasian economy $(m, \boldsymbol{z})$ with an excess demand function $\boldsymbol{z}$ which is λ-Lipschitz-continuous and β-Lipschitz-smooth, a 1-strongly-convex kernel function h, $\alpha \geq 0$, $\eta \in \left(0, \frac{1}{2(2\beta\alpha \text{diam}(\mathcal{X})^2+1+2\lambda)}\right]$, and $\boldsymbol{x}^{(0)} \in \mathcal{X}$.

Consider the mirror potential algorithm (Algorithm 5) run with the regularized primal gap Ξ_α as defined in Equation (5.15), the kernel function h, an arbitrary time horizon $\tau \in \mathbb{N}$, the step size η, the initial iterate $\boldsymbol{x}^{(0)}$, and which outputs $\{\boldsymbol{x}^{(t)}\}_t$. The following convergence bound to a stationary point of Ξ_α then holds:

$$\min_{k=0,1,\ldots,\tau-1} \max_{\boldsymbol{x}\in\mathcal{X}} \langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k)} - \boldsymbol{x} \rangle \leq \frac{2\Xi_\alpha(\boldsymbol{x}^{(0)})}{\tau}$$

In addition, let $\boldsymbol{x}^{(\tau)}_{\text{best}} \in \arg\min_{\boldsymbol{x}^{(k)}:k=0,\ldots,\tau-1} \max_{\boldsymbol{x}\in\mathcal{X}} \langle \nabla \Xi_\alpha(\boldsymbol{x}^{(k)}), \boldsymbol{x}^{(k)} - \boldsymbol{x} \rangle$, then, for some choice of $\tau \in O(\frac{1}{\varepsilon})$, $\boldsymbol{x}^{(\tau)}_{\text{best}}$ is a ε-stationary point of Ξ_α.

Remark 5.5.1 [Walrasian equilibrium under the law of demand and supply]**.**

As pointed out in Remark 4.4.1, when the excess demand is monotone, stationary points of the regularized primal gap function are global solutions. For the case of Walrasian economies, the monotonicity of the excess demand is known under the name of the law of supply and demand, as such to the best of our knowledge this result implies the first polynomial-time computation result for Walrasian equilibrium in general Walrasian economies (i.e., beyond Arrow-Debreu economies) whose excess demand satisfies the law of demand and supply (see, Definition 5.4.8).

Remark 5.5.2 [On Lipschitz-continuity and smoothness]**.**

Recall that we can ensure that the excess demand is Lipschitz-continuous by Lemma 5.4.2 assuming that the price elasticity of the excess demand is bounded. Nevertheless, to the best of our knowledge there exist no market parameters to obtain Lipschitz-smoothness of the excess demand. That said, as the Lipschitz-continuity and Lipschitz-smoothness of a function can be approximated from data, this assumption still remains realistic (see, for instance, (Wood and Zhang, 1996)).

Having established very broad results for the convergence of price-adjustment processes for Walrasian economies, we will now provide an example of a Walrasian economy, namely Fisher markets (Brainard et al., 2000) which have found a great deal of applications in real-world resource allocation problems. This application will also demonstrate that in more restricted Walrasian economies, using a more fine grained analysis, the convergence of *tâtonnement processes* can be guaranteed, hence further complementing the results proven in this chapter.

Chapter 6

Homothetic Fisher Markets: An Example of Walrasian Economy and Tâtonnement

In this chapter, we turn our attention to identifying a large class of Walrasian economies for which the mirror *tâtonnement process* converges to a Walrasian equilibrium. To this end, we make strides towards analyzing the convergence of discrete-time *tâtonnement* in **homothetic Fisher markets**, i.e., Walrasian economies with a fixed supply, and a demand generated by utility-maximizing consumers whose utility functions are given by continuous and homogeneous functions.[1] An important concept in consumer theory is a buyer's Hicksian demand, i.e., consumptions that minimize expenditure while achieving a desired utility level. We identify the maximum elasticity of the Hicksian demand, i.e., the maximum percentage change in the Hicksian demand of any good w.r.t. the change in the price of some other good, as an economic parameter sufficient to capture and explain a range of convergent and non-convergent *tâtonnement* behaviors in a broad class of markets. In particular, we prove the convergence of *tâtonnement* in homothetic Fisher markets with bounded elasticity of Hicksian demand, i.e., Fisher markets in which consumers have preferences represented by homogeneous utility functions for which the elasticity of their Hicksian demand is bounded.

[1]We refer to Fisher markets that comprise buyers with a certain utility function by the name of the utility function, e.g., we call a Fisher market that comprises buyers with Leontief utility functions a Leontief Fisher market. We omit the "continuous" qualifier as Walrasian equilibrium is not guaranteed to exist when utilities are not continuous.

6.1 Background

6.1.1 Mirror Descent

Consider the optimization problem $\min_{\boldsymbol{x} \in V} f(\boldsymbol{x})$, where $f : \mathbb{R}^n \to \mathbb{R}$ is a differentiable convex function and V is the feasible set of solutions. A standard method for solving this problem is the **mirror descent algorithm** (Boyd et al., 2004):

$$\boldsymbol{x}(t+1) = \arg\min_{\boldsymbol{x} \in V} \left\{ \ell_f(\boldsymbol{x}, \boldsymbol{x}(t)) + \gamma_t \text{div}_h(\boldsymbol{x}, \boldsymbol{x}(t)) \right\} \qquad \text{for } t = 0, 1, 2, \dots \tag{6.1}$$

$$\boldsymbol{x}(0) \in \mathbb{R}^n \tag{6.2}$$

Here, $\gamma_t > 0$ is the step size at time t, $\ell_f(\boldsymbol{x}, \boldsymbol{y})$ is the **linear approximation** of f at $\boldsymbol{y}$, that is $\ell_f(\boldsymbol{x}, \boldsymbol{y}) = f(\boldsymbol{y}) + \nabla f(\boldsymbol{y})^T (\boldsymbol{x} - \boldsymbol{y})$, and $\text{div}_h(\boldsymbol{x}, \boldsymbol{x}(t))$ is the **Bregman divergence** of a convex differentiable **kernel** function $h(\boldsymbol{x})$ defined as $\text{div}_h(\boldsymbol{x}, \boldsymbol{y}) = h(\boldsymbol{x}) - \ell_h(\boldsymbol{x}, \boldsymbol{y})$ (Bregman, 1967). In particular, when $h(\boldsymbol{x}) = \frac{1}{2}||\boldsymbol{x}||_2^2$, $\text{div}_h(\boldsymbol{x}, \boldsymbol{y}) = \frac{1}{2}||\boldsymbol{x} - \boldsymbol{y}||_2^2$. In this case, mirror descent reduces to projected gradient descent (Boyd et al., 2004). If instead the kernel is the weighted entropy $h(\boldsymbol{x}) = \sum_{i \in [n]} (x_i \log(x_i) - x_i)$, the Bregman divergence reduces to the **generalized Kullback-Leibler (KL) divergence** (Joyce, 2011):

$$\text{div}_{\text{KL}}(\boldsymbol{x}, \boldsymbol{y}) = \sum_{i \in [n]} \left[x_i \log\left(\frac{x_i}{y_i}\right) - x_i + y_i \right] \quad , \tag{6.3}$$

which, when $V = \mathbb{R}^m_{++}$, yields the following simplified **entropic descent** update rule:

$$\forall j \in [m] \qquad x_j^{(t+1)} = x_j^{(t)} \exp\left\{ \frac{-\mathcal{D}_{x_j} f(\boldsymbol{x}^{(t)})}{\gamma_t} \right\} \qquad \text{for } t = 0, 1, 2, \dots \tag{6.4}$$

$$x_j^{(0)} \in \mathbb{R}_{++} \tag{6.5}$$

A function f is said to be γ**-Bregman-smooth** (Cheung et al., 2018) w.r.t. a Bregman divergence with kernel function h if $f(\boldsymbol{x}) \leq \ell_f(\boldsymbol{x}, \boldsymbol{y}) + \gamma \text{div}_h(\boldsymbol{x}, \boldsymbol{y})$. Birnbaum et al. (2011) showed that if the objective function $f(\boldsymbol{x})$ of a convex optimization problem is γ**-Bregman** w.r.t. to some Bregman divergence div_h, then mirror descent with Bregman divergence div_h converges to an optimal solution $f(\boldsymbol{x}^*)$ at a rate of $O(1/t)$. We require a slightly modified version of this theorem, introduced by Cheung et al. (2013), where it suffices for the γ-Bregman-smoothness property to hold only for consecutive pairs of iterates.

Theorem 6.1.1 [Birnbaum et al. (2011),Cheung et al. (2013)]**.**

Let $\{\boldsymbol{x}^t\}_t$ be the iterates generated by mirror descent with Bregman divergence div_h. Suppose f and h are convex, and for all $t \in \mathbb{N}$ and for some $\gamma > 0$, it holds that $f(\boldsymbol{x}^{(t+1)}) \leq \ell_f(\boldsymbol{x}^{(t+1)}, \boldsymbol{x}^{(t)}) + \gamma \text{div}_h(\boldsymbol{x}^{(t+1)}, \boldsymbol{x}^{(t)})$. If $\boldsymbol{x}^*$ is a minimizer of f, then the following holds for mirror descent with fixed step size γ: for all $t \in \mathbb{N}$, $f(\boldsymbol{x}^{(t)}) - f(\boldsymbol{x}^*) \leq \gamma/t\, \text{div}_h(\boldsymbol{x}^*, \boldsymbol{x}^{(0)})$.

6.1.2 Consumer Theory Primer

Let $\mathcal{X} = \mathbb{R}^m_+$ be a set of possible consumptions over m goods s.t. for any $\boldsymbol{x} \in \mathbb{R}^m_+$ and $j \in [m]$, $x_j \geq 0$ represents the amount of good $j \in [m]$ consumed by consumer (hereafter, buyer) i. The preferences of buyer i over different consumptions of goods can be represented by a **preference relation** $\succeq_i$ over $\mathcal{X}$ such that the buyer (resp. weakly) prefers a choice $\boldsymbol{x} \in \mathcal{X}$ to another choice $\boldsymbol{y} \in \mathcal{X}$ iff $\boldsymbol{x} \succ_i \boldsymbol{y}$ (resp. $\boldsymbol{x} \succeq_i \boldsymbol{y}$). A preference relation is said to be **complete** iff for all $\boldsymbol{x}, \boldsymbol{y} \in \mathcal{X}$, either $\boldsymbol{x} \succeq_i \boldsymbol{y}$ or $\boldsymbol{y} \succeq_i \boldsymbol{x}$, or both. A preference relation is said to be **transitive** if, for all $\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{z} \in \mathcal{X}$, $\boldsymbol{x} \succeq_i \boldsymbol{z}$ whenever $\boldsymbol{x} \succeq_i \boldsymbol{y}$ and $\boldsymbol{y} \succeq_i \boldsymbol{z}$. A preference relation is said to be **continuous** if for any sequence $\{\boldsymbol{x}^{(n)}, \boldsymbol{y}^{(n)}\}_{n \in \mathbb{N}_+} \subset \mathcal{X} \times \mathcal{X}$ $(\boldsymbol{x}^{(n)}, \boldsymbol{y}^{(n)}) \to (\boldsymbol{x}, \boldsymbol{y})$ and $\boldsymbol{x}^{(n)} \succeq_i \boldsymbol{y}^{(n)}$ for all $n \in \mathbb{N}_+$, it also holds that $\boldsymbol{x} \succeq_i \boldsymbol{y}$. A preference relation $\succeq_i$ is said to be **locally non-satiated** iff for all $\boldsymbol{x} \in \mathcal{X}$ and $\epsilon > 0$, there exists $\boldsymbol{y} \in \mathcal{B}_\epsilon(\boldsymbol{x})$ such that $\boldsymbol{y} \succ_i \boldsymbol{x}$. A utility function $u_i : \mathcal{X} \to \mathbb{R}_+$ assigns a positive real value[2] to elements of $\mathcal{X}$, i.e., to all possible consumptions. Every continuous utility function represents some complete, transitive, and continuous preference relation $\succeq_i$ over goods s.t. if $u_i(\boldsymbol{x}) \geq u_i(\boldsymbol{y})$ for two bundles of goods $\boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^m$, then $\boldsymbol{x} \succeq_i \boldsymbol{y}$ (Arrow et al., 1971).

In this paper, we consider the general class of **homothetic** preferences $\succeq_i$ s.t. for any consumption $\boldsymbol{x}, \boldsymbol{y} \in \boldsymbol{X}$ and $\lambda \in \mathbb{R}_+$, $\boldsymbol{x} \succeq_i \boldsymbol{y}$ and $\boldsymbol{y} \succeq_i \boldsymbol{x}$ implies $\lambda \boldsymbol{x} \succeq_i \lambda \boldsymbol{y}$ and $\lambda \boldsymbol{y} \succeq_i \lambda \boldsymbol{x}$, respectively. A preference relation $\succeq_i$ is complete, transitive, continuous, and homothetic iff it can be represented via a continuous and homogeneous utility function u_i of arbitrary degree (Arrow et al., 1971).[3] We note that any homogeneous utility function

[2]Without loss of generality, we assume that utility functions are positive real-valued functions, since any real-valued function can be made positive real-valued by passing it through the monotonic transformation $\boldsymbol{x} \mapsto e^{\boldsymbol{x}}$ without affecting the underlying preference relation.

[3]Throughout this work, without loss of generality, we assume that complete, transitive, continuous, and homothetic preference relations are represented via a homogeneous utility function of degree 1, since any homogeneous utility function of degree k can be made homogeneous of degree 1 without affecting the underlying preference relation by passing the utility function through the monotonic transformation $x \mapsto \sqrt[k]{x}$.

u_i represents locally non-satiated preferences, since for all $\epsilon > 0$ and $\boldsymbol{x} \in \mathcal{X}$, there exists an allocation $(1 + \varepsilon/\|\boldsymbol{x}\|)\boldsymbol{x}$ s.t. $u_i((1 + \varepsilon/\|\boldsymbol{x}\|)\boldsymbol{x}) = (1 + \varepsilon/\|\boldsymbol{x}\|)u_i(\boldsymbol{x}) > u_i(\boldsymbol{x})$, and $[\boldsymbol{x} - (1 + \varepsilon/\|\boldsymbol{x}\|)\boldsymbol{x}] \in \mathcal{B}_\varepsilon(\boldsymbol{x})$.

The class of homogeneous utility functions includes the well-known **constant elasticity of substitution (CES)** utility function family, parameterized by a substitution parameter $-\infty \leq \rho_i \leq 1$, and given by $u_i(\boldsymbol{x}_i) = \sqrt[\rho_i]{\sum_{j \in [m]} v_{ij} x_{ij}^{\rho_i}}$ with each utility function parameterized by the vector of valuations $\boldsymbol{v}_i \in \mathbb{R}_+^n$, where each v_{ij} quantifies the value of good j to buyer i. CES utilities are said to be **gross substitutes** (resp. **gross complements**) CES if $\rho_i > 0$ ($\rho_i < 0$). **Linear utility** functions are obtained when ρ is 1 (goods are perfect substitutes), while **Cobb-Douglas** and **Leontief utility** functions are obtained when $\rho \to 0$ and $\rho \to -\infty$ (goods are perfect complements), respectively:

Linear:	Cobb-Doulas:	Leontief:
$u_i(\boldsymbol{x}_i) = \sum_{j \in [m]} v_{ij} x_{ij}$	$u_i(\boldsymbol{x}_i) = \prod_{j \in [m]} x_{ij}^{v_{ij}}$	$u_i(\boldsymbol{x}_i) = \min_{j : v_{ij} \neq 0} \frac{x_{ij}}{v_{ij}}$

Associated with any consumption $\boldsymbol{x} \in \mathcal{X}$ are **prices** $\boldsymbol{p} \in \mathbb{R}_+^m$ s.t. for all goods $j \in [m]$, $p_j \geq 0$ denotes the price of good j. A **demand correspondence** $\mathcal{F} : \mathbb{R}_+^m \to \mathcal{X}$ takes as input prices $\boldsymbol{p} \in \mathbb{R}_+^m$ and outputs a set of consumptions $\mathcal{F}(\boldsymbol{p})$. If $\mathcal{F}$ is singleton-valued for all $\boldsymbol{p} \in \mathbb{R}_+^m$, then it is called a **demand function**. Given a demand function $\boldsymbol{f}$, we define the **elasticity** $\epsilon_{f_i, x_j} : \mathbb{R}^m \to \mathbb{R}$ of output $f_i(\boldsymbol{x})$ w.r.t. the jth input x_j evaluated at $\boldsymbol{x} = \boldsymbol{y}$ as $\epsilon_{f_i, x_j}(\boldsymbol{y}) = \mathcal{D}_{x_j} f_i(\boldsymbol{y}) \frac{y_j}{f_i(\boldsymbol{y})}$.

A good $j \in [m]$ is said to be a **substitute (resp. complement) w.r.t. a demand function $\boldsymbol{f}$** for a good $k \in [m] \setminus \{j\}$ if the demand $f_j(\boldsymbol{p})$ is increasing (resp. decreasing) in p_k. If a buyer's demand $f_j(\boldsymbol{p})$ for good j is instead weakly increasing (resp. decreasing), good j is said to be a **weak substitute** (resp. **weak complement**) for good k.

Next, we define the **consumer functions** (Mas-Colell et al., 1995; Jehle, 2001). The **indirect utility function** $v_i : \mathbb{R}_+^m \times \mathbb{R}_+ \to \mathbb{R}_+$ takes as input prices $\boldsymbol{p}$ and a budget b_i and outputs the maximum utility the buyer can achieve at that prices within that budget, i.e., $v_i(\boldsymbol{p}, b_i) = \max_{\boldsymbol{x} \in \mathcal{X} : \boldsymbol{p} \cdot \boldsymbol{x} \leq b_i} u_i(\boldsymbol{x})$.

The **Marshallian demand** is a correspondence $\mathcal{D}_i : \mathbb{R}_+^m \times \mathbb{R}_+ \rightrightarrows \mathcal{X}$ that takes as input prices $\boldsymbol{p}$ and a budget b_i and outputs the utility-maximizing allocations of goods at that budget, i.e., $\mathcal{D}_i(\boldsymbol{p}, b_i) = \arg\max_{\boldsymbol{x} \in \mathcal{X} : \boldsymbol{p} \cdot \boldsymbol{x} \leq b_i} u_i(\boldsymbol{x})$.

The **expenditure function** $e_i : \mathbb{R}^m_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ takes as input prices $\boldsymbol{p}$ and a utility level ν_i and outputs the minimum amount the buyer must spend to achieve that utility level at those prices, i.e., $e_i(\boldsymbol{p}, \nu_i) = \min_{\boldsymbol{x} \in \mathcal{X}: u_i(\boldsymbol{x}) \geq \nu_i} \boldsymbol{p} \cdot \boldsymbol{x}$. If the utility function u_i is continuous, then the expenditure function is continuous and homogeneous of degree 1 in $\boldsymbol{p}$ and ν_i jointly, non-decreasing in $\boldsymbol{p}$, strictly increasing in ν_i, and concave in $\boldsymbol{p}$.

The **Hicksian demand** is a correspondence $\mathcal{H}_i : \mathbb{R}^m_+ \times \mathbb{R}_+ \rightrightarrows \mathbb{R}_+$ that takes as input prices $\boldsymbol{p}$ and a utility level ν_i and outputs the cost-minimizing allocations of goods at those prices and utility level, i.e., $\mathcal{H}_i(\boldsymbol{p}, \nu_i) = \arg\min_{\boldsymbol{x} \in \mathcal{X}: u_i(\boldsymbol{x}) \geq \nu_i} \boldsymbol{p} \cdot \boldsymbol{x}$.

6.1.3 Fisher Markets

A **Fisher market** $(n, m, \boldsymbol{u}, \boldsymbol{b})$, denoted $(\boldsymbol{u}, \boldsymbol{b})$ when clear from context, consists of n traders and m goods. A **Fisher market** consists of n buyers and m divisible goods (Brainard et al., 2000). Each buyer $i \in [n]$ has a budget $b_i \in \mathbb{R}_{++}$ and a utility function $u_i : \mathbb{R}^m_+ \to \mathbb{R}$. As is standard in the literature, we assume there is one unit of each good, and one unit of currency available in the market, i.e. $\sum_{i \in [n]} b_i = 1$ (Nisan and Roughgarden, 2007). We denote $\boldsymbol{u} = (u_1, \ldots, u_n)$, and $\boldsymbol{b} \doteq (b_1, \ldots, b_n)$.

An **allocation** $\boldsymbol{X}$ is a map from goods to buyers, represented as a matrix s.t. $x_{ij} \geq 0$ denotes the amount of good $j \in [m]$ allocated to buyer $i \in [n]$. Goods are assigned **prices** $\boldsymbol{p} \in \mathbb{R}^m_+$.

When the buyers' utility functions in a Fisher market are all of the same type, we qualify the market by the name of the utility function, e.g., a linear Fisher market. A **mixed CES Fisher market** is a Fisher market which comprises CES buyers with possibly different substitution parameters. Considering properties of goods, rather than buyers, a (Fisher) market satisfies **gross substitutes** (resp. **gross complements**) if all pairs of goods in the market are gross substitutes (resp. gross complements). We define net substitute Fisher markets and net complements Fisher markets similarly. We refer the reader to Figure 3.1a for a summary of the relationships among various Fisher markets.

Definition 6.1.1 [Fisher Equilibrium]**.**

A tuple $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ is said to be a **Fisher equilibrium** of a Fisher market $(\boldsymbol{u}, \boldsymbol{b})$ iff

(Utility maximization) Buyers maximize their utility constrained by their budget, i.e., $\forall i \in [n], \boldsymbol{x}_i^* \in \mathcal{D}_i(\boldsymbol{p}^*, b_i)$;

(Feasibility) $\forall j \in [m], \sum_{i \in [n]} x_{ij}^* \leq 1$

(Walras' law) $\boldsymbol{p}^* \cdot \left(\sum_{i \in [n]} \boldsymbol{x}_i^* - \mathbf{1}_m \right)$

Any Fisher market $(\boldsymbol{u}, \boldsymbol{b})$ can be represented as a Walrasian economy. To this end, overloading notation, define the **aggregate demand correspondence** $\mathcal{D} : \mathbb{R}_+^m \rightrightarrows \mathbb{R}_+^m$ at prices $\boldsymbol{p}$ as the sum of the Marshallian demand at $\boldsymbol{p}$, given budgets $\boldsymbol{b}$, i.e., $\mathcal{D}(\boldsymbol{p}) = \sum_{i \in [n]} \mathcal{D}_i(\boldsymbol{p}, b_i)$. The **excess demand correspondence** $\mathcal{Z} : \mathbb{R}^m \rightrightarrows \mathbb{R}^m$ of a Fisher market $(\boldsymbol{u}, \boldsymbol{b})$, which takes as input prices and outputs a set of excess demands at those prices, Any Fisher market $(\boldsymbol{u}, \boldsymbol{b})$ can be represented as a Walrasian economy $(m, \mathbb{R}_+^m, \mathcal{Z})$ where $\mathcal{Z}$ is defined as the difference between the aggregate demand for and the supply of each good: i.e., $\mathcal{Z}(\boldsymbol{p}) = \mathcal{D}(\boldsymbol{p}) - \mathbf{1}_m$ where $\mathbf{1}_m$ is the vector of ones of size m, and $\mathcal{D}(\boldsymbol{p}) - \mathbf{1}_m = \{\boldsymbol{x} - \mathbf{1}_m \mid \forall \boldsymbol{x} \in \mathcal{D}(\boldsymbol{p})\}$. Note that the the set of Fisher equilibrium prices of any Fisher market $(\boldsymbol{u}, \boldsymbol{b})$ is equal to the set of Walrasian equilibria of $(m, \mathcal{Z})$.

6.2 Homothetic Fisher Markets

We now turn our attention to the computation of Fisher equilibrium in Fisher markets. We will restrict ourselves to a large class of Fisher markets, namely homothetic Fisher markets.

Definition 6.2.1 [Homothetic Fisher Markets]**.**
A **homothetic Fisher market** is a Fisher market $(\boldsymbol{u}, \boldsymbol{b})$ s.t. for each buyer $i \in [n]$:

(Continuity) u_i is continuous;

(Homothetic preferences) u_i is homogeneous, i.e., for all $\lambda \geq 0$, $\lambda u_i(\boldsymbol{x}_i) = u_i(\lambda \boldsymbol{x}_i)$.

Suppose that $(\boldsymbol{u}, \boldsymbol{b})$ is a continuous, concave, and homogeneous Fisher market. The optimal solutions $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ to the primal and dual of **Eisenberg-Gale program** constitute a Fisher equilibrium of $(\boldsymbol{u}, \boldsymbol{b})$

(Devanur et al., 2002; Eisenberg and Gale, 1959; Jain et al., 2005):[4]

Primal

$$\max_{\boldsymbol{X} \in \mathbb{R}_+^{n \times m}} \sum_{i \in [n]} b_i \log\left(u_i(\boldsymbol{x}_i)\right)$$

$$\text{subject to} \quad \sum_{i \in [n]} x_{ij} \leq 1 \qquad \forall j \in [m]$$

Dual

$$\min_{\boldsymbol{p} \in \Delta_m} \sum_{j \in [m]} p_j + \sum_{i \in [n]} \left[b_i \log\left(v_i(\boldsymbol{p}, b_i)\right) - b_i\right]$$

We now introduce a convex program which is equivalent to the Eisenberg-Gale convex program, but whose optimal value differs from that of the Eisenberg-Gale convex program by an additive constant. Before presenting our program, we present several preliminary lemmas. All omitted proofs can be found in Section 7.2.

The next lemma establishes an important property of the indirect utility and expenditure functions in CCH Fisher markets that we heavily exploit in this work, namely that the derivative of the indirect utility function with respect to b_i—the bang-per-buck—is constant across all budget levels. Likewise, the derivative of the expenditure function with respect to ν_i—the buck-per-bang—is constant across all utility levels. In other words, both functions effectively depend only on prices. Not only are the bang-per-buck and the buck-per-bang constant, they equal $v_i(\boldsymbol{p}, 1)$ and $e_i(\boldsymbol{p}, 1)$, respectively, namely their values at exactly one unit of budget and one unit of (indirect) utility.

An important consequence of this lemma is that, by picking prices that maximize a buyer's bang-per-buck, we not only maximize their bang-per-buck at all budget levels, but we further maximize their total indirect utility, given their *known* budget. In particular, given prices $\boldsymbol{p}^*$ that maximize a buyer's bang-per-buck at budget level 1, we can easily calculate the buyer's total (indirect) utility at budget b_i by simply multiplying their bang-per-buck by b_i: i.e., $v_i(\boldsymbol{p}^*, b_i) = b_i v_i(\boldsymbol{p}^*, 1)$. Here, we see quite explicitly the homogeneity assumption at work.

Analogously, by picking prices that maximize a buyer's buck-per-bang, we not only maximize their buck-per-bang at all utility levels, but we further maximize the buyer's total expenditure, given their *unknown* optimal utility level. As above, given prices $\boldsymbol{p}^*$ that minimize a buyer's buck-per-bang at utility level 1, we

[4]The dual as presented here was formulated by Goktas et al. (2022).

can easily calculate the buyer's total expenditure at utility level ν_i by simply multiplying their buck-per-bang by ν_i: i.e., $e_i(\boldsymbol{p}^*, \nu_i) = \nu_i e_i(\boldsymbol{p}^*, 1)$.

In sum, solving for optimal prices at any budget level, or analogously at any utility level, requires only a single optimization, in which we solve for optimal prices at budget level, or utility level, 1.

Lemma 6.2.1.

If u_i is continuous and homogeneous of degree 1, then $v_i(\boldsymbol{p}, b_i)$ and $e_i(\boldsymbol{p}, \nu_i)$ are differentiable in b_i and ν_i, resp. Further, $\mathcal{D}_{b_i} v_i(\boldsymbol{p}, b_i) = \{v_i(\boldsymbol{p}, 1)\}$ and $\mathcal{D}_{\nu_i} e_i(\boldsymbol{p}, \nu_i) = \{e_i(\boldsymbol{p}, 1)\}$.

The next lemma provides further insight into why CCH Fisher markets are easier to solve than non-CCH Fisher markets. The lemma states that the bang-per-buck, i.e., the marginal utility of an additional unit of budget, is equal to the inverse of its buck-per-bang, i.e., the marginal cost of an additional unit of utility. Consequently, by setting prices so as to minimize the buck-per-bang of buyers, we can also maximize their bang-per-buck. Since the buck-per-bang is a function of prices only, and not of prices and allocations together, this lemma effectively decouples the calculation of equilibrium prices from the calculation of equilibrium allocations, which greatly simplifies the problem of computing Fisher equilibria in CCH Fisher markets.

Corollary 6.2.1.

If buyer i's utility function u_i is CCH, then

$$\frac{1}{e_i(\boldsymbol{p}, 1)} = \frac{1}{\frac{\partial e_i(\boldsymbol{p}, \nu_i)}{\partial \nu_i}} = \frac{\partial v_i(\boldsymbol{p}, b_i)}{\partial b_i} = v_i(\boldsymbol{p}, 1) \quad . \tag{6.6}$$

We can now present our characterization of the dual of the Eisenberg-Gale program via expenditure functions. While Devanur et al. (Devanur et al., 2016) provided a method to construct a similar program to that given in Theorem 6.2.1 for specific utility functions, their method does not apply to arbitrary CCH utility functions. The proof of this theorem can be found in Section 7.2.

Theorem 6.2.1 [New Convex Program for Homothetic Fisher Markets]**.**

The optimal solutions $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ to the following primal and dual convex programs correspond to Fisher equilibrium allocations and prices, respectively, of the homothetic Fisher market $(\boldsymbol{u}, \boldsymbol{b})$:

Primal

$$\max_{\boldsymbol{X} \in \mathbb{R}_+^{n \times m}} \quad \sum_{i \in [n]} \left[b_i \log u_i \left(\frac{\boldsymbol{x}_i}{b_i} \right) + b_i \right]$$

$$\text{subject to} \quad \sum_{i \in [n]} x_{ij} \leq 1 \qquad \forall j \in [m]$$

Dual

$$\min_{\boldsymbol{p} \in \Delta_m} \psi(\boldsymbol{p}) \doteq \sum_{j \in [m]} p_j - \sum_{i \in [n]} b_i \log \left(e_i(\boldsymbol{p}, 1) \right)$$

Our new convex program for CCH Fisher markets makes plain the duality structure between utility functions and expenditure functions that is used to compute "shadow" prices for allocations. In particular, $e_i(\boldsymbol{p}, \nu_i)$ is the Fenchel conjugate of the indicator function $\chi_{\{\boldsymbol{x}: u_i(\boldsymbol{x}_i) \geq \nu_i\}}$, meaning the utility levels and expenditures are dual (in a colloquial sense) to one another. Therefore, equilibrium utility levels can be determined from equilibrium expenditures, and vice-versa, which implies that allocations and prices can likewise be derived from one another through this duality structure.[5]

Since the objective function of the primal in Theorem 6.2.1 is in general non-concave (i.e., if utilities $\boldsymbol{u}$ are not concave), strong duality need not hold; however, the dual is still guaranteed to be convex (Boyd et al., 2004). This observation suggests that even if the problem of computing Fisher equilibrium *allocations* is non-concave, the problem of computing Fisher equilibrium *prices* can still be convex. Additionally, since this convex program differs from the Eisenberg-Gale program by an additive constant, we obtain as a corollary that solutions to the Einseberg-Gale program also correspond to Fisher equilibria in *all* homothetic Fisher markets, including those in which the buyers' utility functions are non-concave.

6.3 Convex Potential Markets

An interesting property of this convex program is that its dual expresses Fisher equilibrium prices via expenditure functions, and just like the Eisenberg-Gale program's dual objective (Cheung et al., 2013; Devanur et al., 2008), the gradient of its objective $\psi(\boldsymbol{p})$ at any price $\boldsymbol{p}$ is equal to the negative excess demand in the market at those prices.

Cheung et al. (Cheung et al., 2013) showed via the Lagrangian of the Eisenberg-Gale program, i.e., without constructing the precise dual, that the subdifferential of the dual of the Eisenberg-Gale program is equal to

[5] A more in-depth analysis of this duality structure can be found in Blume (Blume, 2017).

the negative excess demand in the associated market, which implies that mirror descent equivalent to a subset of tâtonnement rules. In this section, we use a generalization of Shephard's lemma to prove that the subdifferential of the dual of our new convex program is equal to the negative excess demand in the associated market. Our proof also applies to the dual of the Eisenberg-Gale program, since the two duals differ only by a constant factor.

Shephard's lemma tells us that the rate of change in expenditure with respect to prices, evaluated at prices $\boldsymbol{p}$ and utility level ν_i, is equal to the Hicksian demand at prices $\boldsymbol{p}$ and utility level ν_i. Alternatively, the partial derivative of the expenditure function with respect to the price p_j of good j at utility level ν_i is simply the share of the total expenditure allocated to j divided by the price of j, which is exactly the Hicksian demand for j at utility level ν_i.

While Shephard's lemma is applicable to utility functions with singleton-valued Hicksian demand (i.e., strictly concave utility functions), we require a generalization of Shephard's lemma that applies to utility functions that could have set-valued Hicksian demand. An early proof of this generalized lemma was given by Tanaka in a discussion paper (Tanaka, 2008); a more modern perspective can be found in a recent survey by Blume (Blume, 2017). For completeness, we also provide a new, simple proof of this result via Danskin's theorem (for subdifferentials) in (Danskin, 1966) Section 7.2.

Lemma 6.3.1 [Shephard's lemma, generalized for set-valued Hicksian demand (Blume, 2017; Shephard, 2015; Tanaka, 2008)]**.**
Let $e_i(\boldsymbol{p}, \nu_i)$ be the expenditure function of buyer i and $\boldsymbol{h}_i(\boldsymbol{p}, \nu_i)$ be the Hicksian demand set of buyer i. The subdifferential $\mathcal{D}_{\boldsymbol{p}} e_i(\boldsymbol{p}, \nu_i)$ is the Hicksian demand at prices $\boldsymbol{p}$ and utility level ν_i, i.e., $\mathcal{D}_{\boldsymbol{p}} e_i(\boldsymbol{p}, \nu_i) = \boldsymbol{h}_i(\boldsymbol{p}, \nu_i)$.

The next lemma plays an essential role in the proof that the subdifferential of the dual of our convex program is equal to the negative excess demand. Just as Shephard's Lemma related the expenditure function to Hicksian demand via (sub)gradients, this lemma relates the expenditure function to Marshallian demand via (sub)gradients. One way to understand this relationship is in terms of **Marshallian consumer surplus**, the area under the Marshallian demand curve, i.e., the integral of Marshallian demand with respect to prices.[6] Specifically, by applying the fundamental theorem of calculus to the left-hand side of Lemma 6.3.2,

[6]We note that the definition of Marshallian consumer surplus for multiple goods requires great care and falls outside the scope of this paper. More information on consumer surplus can be found in Levin (Levin, 2004), and Vives (Vives, 1987).

we see that the Marshallian consumer surplus equals $b_i \log\left(\frac{\partial e_i(\boldsymbol{p}, \nu_i)}{\partial \nu_i}\right)$. The key takeaway is thus that any objective function we might seek to optimize that includes a buyer's Marshallian consumer surplus is thus optimizing their Marshallian demand, so that optimizing this objective yields a utility-maximizing allocation for the buyer, constrained by their budget.

Lemma 6.3.2.

If buyer i's utility function u_i is continuous and homogeneous, then $\mathcal{D}_{\boldsymbol{p}}\left(b_i \log\left(\frac{\partial e_i(\boldsymbol{p}, \nu_i)}{\partial \nu_i}\right)\right) = \boldsymbol{d}_i(\boldsymbol{p}, b_i)$.

Remark 6.3.1.

Lemma 6.3.2 makes the dual of our convex program easy to interpret, and thus sheds light on the dual of the Eisenberg-Gale program. Specifically, we can interpret the dual as specifying prices that minimize the distance between the sellers' surplus and the buyers' Marshallian surplus. The left hand term is simply the sellers' surplus, and by Lemma 6.3.2, the right hand term can be seen as the buyers' total Marshallian surplus.

Remark 6.3.2.

The lemmas we have proven in this section and the last provide a possible explanation as to why no primal-dual type convex program is known that solves Fisher markets when buyers have *non-homogeneous* utility functions, in which the primal describes optimal allocations while the dual describes equilibrium prices. By the homogeneity assumption, a CCH buyer can increase their utility level (resp. decrease their spending) by $c\%$ by increasing their budget (resp. decreasing their desired utility level) by $c\%$ (Lemma 7.2.1). This observation implies that the marginal expense of additional utility, i.e., "bang-per-buck", and the marginal utility of additional budget, i.e., "buck-per-bang", are constant (Lemma 6.2.1). Additionally, optimizing prices to maximize buyers' "bang-per-buck" is equivalent to optimizing prices to minimize their "buck-per-bang" (Corollary 6.2.1). Further, optimizing prices to minimize their "buck-per-bang" is equivalent to maximizing their utilities constrained by their budgets (Lemma 6.3.2). Thus, the equilibrium prices computed by the dual of our program, which optimize the buyers' buck-per-bang, simultaneously optimize their utilities constrained by their budgets. In particular, equilibrium prices can be computed without reference to equilibrium allocations (Corollary 6.2.1 + Lemma 6.3.2). In other words, assuming homogeneity, the computation of the equilibrium allocations and prices can be isolated into separate primal and dual problems.

Next, we show that the subdifferential of the dual of our convex program is equal to the negative excess demand in the associated market.

Theorem 6.3.1.

Given any homothetic Fisher market $(\boldsymbol{u}, \boldsymbol{b})$, the subdifferential of the dual of the program in Theorem 6.2.1 at any price $\boldsymbol{p}$ is equal to the negative excess demand in $(\boldsymbol{u}, \boldsymbol{b})$ at price $\boldsymbol{p}$: i.e., $\mathcal{D}_{\boldsymbol{p}}\psi(\boldsymbol{p}) = -\mathcal{Z}(\boldsymbol{p})$.

Cheung et al. (Cheung et al., 2013) define a class of markets called **convex potential function (CPF)** markets. A market is a CPF market, if there exists a convex potential function φ such that $\mathcal{D}_{\boldsymbol{p}}\varphi(\boldsymbol{p}) = -\boldsymbol{z}(\boldsymbol{p})$. They then prove that Fisher markets are CPF markets by showing, through the Lagrangian of the Eisenberg-Gale program, that its dual is a convex potential function (Cheung et al., 2013). Likewise, Theorem 6.3.1 implies the following:

Corollary 6.3.1.

All homothetic Fisher markets are CPF markets.

Proof

A convex potential function $\phi : \mathbb{R}^m \to \mathbb{R}$ for any CCH Fisher market $(\boldsymbol{u}, \boldsymbol{b})$ is given by:

$$\psi(\boldsymbol{p}) = \sum_{j\in[m]} p_j - \sum_{i\in[n]} b_i \log\left(\frac{\partial e_i(\boldsymbol{p}, \nu_i)}{\partial \nu_i}\right) \tag{6.7}$$

6.4 Market Parameters

An important consequence of the fact that implies that mirror descent on φ over the positive ortanth is equivalent to *tâtonnement* in all homothetic Fisher markets. Using this equivalence, we can pick a particular kernel function h, and then potentially use Theorem 6.1.1 to establish convergence rates for *tâtonnement*.

Unfortunately, *tâtonnement* does not converge to equilibrium prices in all homothetic Fisher markets, e.g., linear Fisher markets (Cole and Tao, 2019), which suggests the need for additional restrictions on the class of homothetic Fisher markets. Goktas et al. (2022) suggest the maximum absolute value of the Marshallian price demand elasticity, i.e., $c = \max_{j,k,i} \max_{(\boldsymbol{p},\boldsymbol{b}) \in \Delta_m \times \Delta_n \times [n]} \left\| \epsilon_{d_{ij},p_k}(\boldsymbol{p}, b_i) \right\|$, as a possible market parameter to use to establish a convergence rate of $O(^{(1+c)}/_T)$. However, Cole and Fleischer's [2008] results suggest that it is unlikely that Marshallian demand elasticity could be enough, since the proof techniques used in work that makes this assumption require one to quantify the direction of the change in demand as a function of the change in the prices of the other goods, and hence only apply when one assumes WGS or WGC (Cole and Fleischer, 2008).

One of the main contributions of this paper is the observation that the maximum absolute value of the price elasticity of Hicksian demand in a homothetic Fisher market is sufficient to analyze the convergence of *tâtonnement*. To this end, in this section, we anaylyze Hicksian demand price elasticity, exposit some of its properties in homothetic markets, and argue why it is a natural parameter to consider in the analysis of *tâtonnement*.

We first note that for Leontief utilities, the Hicksian cross-price elasticity of demand is equal to 0, while for linear utilities the Hicksian cross-price elasticity of demand is, by convention, ∞.[7] For Cobb-Douglas utilities, the Hicksian cross-price elasticity of demand is strictly positive and upper bounded by 1, but it is not the same for all pairs of goods. Note that the behavior of the Hicksian cross-price elasticity of demand is radically different than that of the Marshallian cross-price elasticity of demand, for which the elasticities of linear, Cobb-Douglas, and Leontief utilities are respectively given as ∞, 0, and $-\infty$. A taxonomy of utility classes as a function of price elasticity of demand (both Marshallian and Hicksian) is shown in Figure 3.1b (Section 3.3).

We start our analysis with following lemma, which shows that the Hicksian price elasticity of demand is constant across all utility levels in homothetic Fisher markets. This property implies that the Hicksian demand price elasticity at one unit of utility provides sufficient information about the market's reactivity to changes in prices, even without any information about the buyers' utility levels. This information is crucial

[7] The limit of Hicksian price elasticity of demand as $\rho \to 1$ is not well defined, i.e., if $\rho \to 1^-$ the limit is $+\infty$, while if $\rho \to 1^+$ the limit is $-\infty$. However, for linear utilities, as the Hicksian demand for a good can only go up when the price of another good goes up, we set the elasticity of Hicksian price elasticity of demand for linear utilities to be $+\infty$, by convention. We refer the reader to Ramskov and Munksgaard (2001) for a primer on elasticity of demand.

when trying to bound the changes in Hicksian demand from one iteration of *tâtonnement* to another, since buyers' utilities can change.[8]

Lemma 6.4.1.

For any Hicksian demand $\boldsymbol{h}_i$ associated with a homogeneous utility function u_i, for all $j, k \in [m], \boldsymbol{p} \in \mathbb{R}_+^m, \nu_i \in \mathbb{R}_+$, it holds that $\epsilon_{h_{ij},p_k}(\boldsymbol{p}, \nu_i) = \epsilon_{h_{ij},p_k}(\boldsymbol{p}, 1) = 1$.

With the above lemma in hand, we now explain why the Hicksian demand price elasticity[9] is a better market parameter by which to analyze the convergence of *tâtonnement* than the Marshallian demand price elasticity. Cheung et al. (2013); Cheung (2014) use the dual of the Eisenberg-Gale program as a potential to measure the progress that *tâtonnement* makes at each step, for (nested) CES and Leontief utilities. Under these functional forms, the authors are able to explain a change in the value of the buyers' indirect utilities as a function a change in prices, based on which they bound the change in the second term of the dual $\sum_{i\in[n]} b_i \log(v_i(\boldsymbol{p}, b_i)) - b_i$ from one time period to the next. Using this bound, they show that *tâtonnement* makes steady progress towards equilibrium.

However, in general homothetic Fisher markets, knowing how much the Marshallian demand for each good changes from one iteration of *tâtonnement* to another does not tell us how much the buyers' utilities change. More concretely, suppose that the Marshallian demand of a buyer i has changed by an additive vector $\Delta\boldsymbol{d}_i$ from time t to time $t+1$, then the difference in indirect utilities from one period to another is given by $u_i(\boldsymbol{d}_i^{(t+1)}) - u_i(\boldsymbol{d}_i^{(t)}) = u_i(\boldsymbol{d}_i^{(t)} + \Delta\boldsymbol{d}_i) - u_i(\boldsymbol{d}_i^{(t)})$. Without additional information about the utility functions, e.g., Lipschitz continuity, it is impossible to bound this difference, because utilities can change by an unbounded amount from one period to another. Hence, even if the Marshallian price elasticity of demand and the changes in prices from one period to another were known, it would only allow us to bound the difference in demands, and not the difference in indirect utilities. To get around this difficulty, one could consider making an assumption about the boundedness of the indirect utility function's price elasticity, or the utility function's Lipschitz-continuity, but such assumptions would not be economically justified, since

[8]We include all omitted results and proofs in Section 7.2.

[9]Going forward we refer to the Hicksian price elasticity of demand, as simply Hicksian demand elasticity, because Hicksian price elasticity of demand w.r.t. utility level is always 1.

utility functions are merely representations of preference orderings without any inherent meaning of their own.

We can circumvent this issue by instead looking at the dual of the convex program in Theorem 6.2.1. In this dual, the indirect utility term is replaced by the expenditure function. The advantage of this formulation is that if one knows the amount by which prices change from one iteration to the next, as well as the Hicksian demand elasticity, then we can easily bound the change in spending from one period to another.

The following lemma is crucial to proving the convergence of *tâtonnement*. This lemma allows us to bound the changes in buyer spending across all time periods, thereby allowing us to obtain a global convergence rate. In particular, it shows that the change in spending between two consecutive iterations of *tâtonnement* can be bounded as a function of the prices and the Hicksian demand elasticity.

More formally, suppose that we would like to bound the percentage change in expenditure at one unit of utility from one iteration to another, i.e., $\frac{e_i(\boldsymbol{p}^{(t+1)},1)-e_i(\boldsymbol{p}^{(t)},1)}{e_i(\boldsymbol{p}^{(t)},1)}$, using a first order Taylor expansion of $e_i(\boldsymbol{p}^{(t)} + \Delta\boldsymbol{p}, 1)$ around $\boldsymbol{p}^{(t)}$. By Taylor's theorem (Graves, 1927), we have: $e_i(\boldsymbol{p}^{(t)} + \Delta\boldsymbol{p}, 1) = e_i(\boldsymbol{p}^{(t)}, 1) + \left\langle \nabla_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)}, 1), \Delta\boldsymbol{p} \right\rangle + 1/2 \left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1)\Delta\boldsymbol{p}, \Delta\boldsymbol{p} \right\rangle$ for some $c \in (0, 1)$. Re-organizing terms around, we get $e_i(\boldsymbol{p}^{(t+1)}, 1) - e_i(\boldsymbol{p}^{(t)}, 1) = \left\langle \nabla_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)}, 1), \Delta\boldsymbol{p} \right\rangle + 1/2 \left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1)\Delta\boldsymbol{p}, \Delta\boldsymbol{p} \right\rangle$. Dividing both sides by $e_i(\boldsymbol{p}^{(t)}, 1)$ we obtain:

$$\frac{\left\langle \nabla_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)}, 1), \Delta\boldsymbol{p} \right\rangle + 1/2 \left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1)\Delta\boldsymbol{p}, \Delta\boldsymbol{p} \right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)}$$

We can now apply Shephard's lemma (Shephard, 2015), a corollary of the envelope theorem (Afriat, 1971; Milgrom and Segal, 2002), to the numerator, which allows us to conclude that for all buyers $i \in [n]$, $\nabla_{\boldsymbol{p}} e_i(\boldsymbol{p}, \nu_i) = \boldsymbol{h}_i(\boldsymbol{p}, \nu_i)$. Next, using the definition of the expenditure function in the denominator, we obtain the following:

$$= \frac{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \Delta\boldsymbol{p} \right\rangle}{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t)} \right\rangle} + \frac{1/2 \left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1)\Delta\boldsymbol{p}, \Delta\boldsymbol{p} \right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)} \tag{6.8}$$

If the change in prices is bounded, and the Hicksian demand elasticity is known, then one can bound the first term in Equation (6.8) with ease. It remains to be seen if the second term can be bounded. The following lemma provides an affirmative answer to that question. In particular, we show that the second-order error term in the Taylor approximation above can be bounded as a function of the maximum absolute value of the

Hicksian demand elasticity. We note that in the following lemma, by Lemma 7.2.9, the Marshallian demand is unique, because the Hicksian demand is a singleton for bounded elasticity of Hicksian demand.

Lemma 6.4.2.

Fix $i \in [n]$ and $t \in \mathbb{N}_+$ and let $\Delta \boldsymbol{p} = \boldsymbol{p}^{(t+1)} - \boldsymbol{p}^{(t)}$. Suppose that $\frac{|\Delta p_j|}{p_j^{(t)}} \leq \frac{1}{4}$, then for all buyers $i \in [n]$, and for some $c \in (0,1)$, it holds that:

$$\left| \frac{b_i}{2} \frac{\left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1)\Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)} \right| \leq \frac{5\epsilon}{6} \sum_j \frac{(\Delta p_j)^2}{p_j^{(t)}} d_{ij}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i) \ , \tag{6.9}$$

where $\epsilon \doteq \max_{\boldsymbol{p} \in \Delta_m, j,k \in [m]} \left| \epsilon_{h_{ij}, p_k}(\boldsymbol{p}, 1) \right|$.

Proof of Lemma 6.4.2

By Shephard's lemma (Shephard, 2015) (Lemma 6.3.1, Section 7.2), it holds that $\left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1)\Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle = \left\langle \nabla_{\boldsymbol{p}} \boldsymbol{h}_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1)\Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle$.

$$\begin{aligned}
&\left| \frac{b_i}{2} \frac{\left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1)\Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle}{\left\langle \boldsymbol{h}_i^t, \boldsymbol{p}^{(t)} \right\rangle} \right| \\
&= \left| \frac{b_i}{2} \frac{\left\langle \nabla_{\boldsymbol{p}} \boldsymbol{h}_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1)\Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle}{\left\langle \boldsymbol{h}_i^t, \boldsymbol{p}^{(t)} \right\rangle} \right| && \text{(Shephard's Lemma)} \\
&\leq \frac{b_i}{2} \frac{\sum_{j,k} \left| \Delta p_j \right| \left| \mathcal{D}_{p_k} h_{ij}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \right| \left| \Delta p_k \right|}{\left\langle \boldsymbol{h}_i^t, \boldsymbol{p}^{(t)} \right\rangle} \\
&= \frac{b_i}{2} \frac{\sum_{j,k} \left| \Delta p_j \right| \sqrt{\left| \mathcal{D}_{p_k} h_{ij}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \right|} \sqrt{\left| \mathcal{D}_{p_k} h_{ij}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \right|} \left| \Delta p_k \right|}{\left\langle \boldsymbol{h}_i^t, \boldsymbol{p}^{(t)} \right\rangle} \\
&= \frac{b_i}{2} \frac{\sum_{j,k} \left| \Delta p_j \right| \sqrt{\left| \mathcal{D}_{p_j} h_{ik}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \right|} \sqrt{\left| \mathcal{D}_{p_k} h_{ij}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \right|} \left| \Delta p_k \right|}{\left\langle \boldsymbol{h}_i^t, \boldsymbol{p}^{(t)} \right\rangle} && (6.10)
\end{aligned}$$

where the last was obtained from the symmetry of $\nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}, \nu_i) = \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}, \nu_i)^T$ for all $i \in [n], \boldsymbol{p} \in \mathbb{R}^m_+, \nu_i \in \mathbb{R}_+$ (Mas-Colell et al., 1995), which combined with Shephard's lemma gives us $\nabla_{\boldsymbol{p}} \boldsymbol{h}_i(\boldsymbol{p}, \nu_i) = \nabla_{\boldsymbol{p}} \boldsymbol{h}_i(\boldsymbol{p}, \nu_i)^T$, i.e., for all $j,k \in [m]$, $\mathcal{D}_{p_j} h_{ik}(\boldsymbol{p}, \nu_i) = \mathcal{D}_{p_k} h_{ij}(\boldsymbol{p}, \nu_i)$.

Define the Hicksian demand elasticity of buyer i for good j w.r.t. the price of good k as $\epsilon_{h_{ij}, p_k}(\boldsymbol{p}, \nu_i) = \mathcal{D}_{p_k} h_{ij}(\boldsymbol{p}, \nu_i) \frac{p_k}{h_{ij}(\boldsymbol{p}, \nu_i)}$. Since utility functions are homogeneous, by Lemma 6.4.1 we have for all $\nu_i \in \mathbb{R}_+$, $\epsilon_{h_{ij}, p_k}(\boldsymbol{p}, \nu_i) = \mathcal{D}_{p_k} h_{ij}(\boldsymbol{p}, \nu_i) \frac{p_k}{h_{ij}(\boldsymbol{p}, \nu_i)} = \mathcal{D}_{p_k} h_{ij}(\boldsymbol{p}, 1) \frac{p_k}{h_{ij}(\boldsymbol{p}, 1)}$. Re-organizing expressions, we get $\mathcal{D}_{p_k} h_{ij}(\boldsymbol{p}, 1) = \epsilon_{h_{ij}, p_k}(\boldsymbol{p}, 1) \frac{h_{ij}(\boldsymbol{p}, 1)}{p_k}$. Going back to Equation (6.10), we get:

$$= \frac{b_i}{2} \frac{\sum_{j,k} \left|\Delta p_j\right| \sqrt{\left|\epsilon_{h_{ik},p_j}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1)\frac{h_{ik}(\boldsymbol{p}^{(t)}+c\Delta\boldsymbol{p},1)}{p_j^{(t)}+c\Delta p_j}\right|} \sqrt{\left|\epsilon_{h_{ij},p_k}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1)\frac{h_{ij}(\boldsymbol{p}^{(t)}+c\Delta\boldsymbol{p},1)}{p_k^{(t)}+c\Delta p_k}\right|} \left|\Delta p_k\right|}{\left\langle \boldsymbol{h}_i^t, \boldsymbol{p}^{(t)} \right\rangle}$$

$$= \frac{b_i}{2} \frac{\sum_{j,k} \left|\Delta p_j\right| \sqrt{\left|\epsilon_{h_{ik},p_j}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1)\right| \frac{h_{ik}(\boldsymbol{p}^{(t)}+c\Delta\boldsymbol{p},1)}{p_j^{(t)}+c\Delta p_j}} \sqrt{\left|\epsilon_{h_{ij},p_k}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1)\right| \frac{h_{ij}(\boldsymbol{p}^{(t)}+c\Delta\boldsymbol{p},1)}{p_k^{(t)}+c\Delta p_k}} \left|\Delta p_k\right|}{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t)} \right\rangle}$$

Letting $\epsilon = \max_{\boldsymbol{p}\in\mathbb{R}_+^m, \nu_i \in \mathbb{R}_+, j,k\in[m]} \left|\epsilon_{h_{ij},p_k}(\boldsymbol{p}, \nu_i)\right|$. Note that since utility functions are homogeneous, by Lemma 6.4.1 we have $\epsilon = \max_{\boldsymbol{p}\in\mathbb{R}_+^m, \nu_i \in \mathbb{R}_+, j,k\in[m]} \left|\epsilon_{h_{ij},p_k}(\boldsymbol{p}, \nu_i)\right| = \max_{\boldsymbol{p}\in\mathbb{R}_+^m, j,k\in[m]} \left|\epsilon_{h_{ij},p_k}(\boldsymbol{p}, 1)\right|$, which gives us:

$$\leq \frac{\epsilon b_i}{2} \frac{\sum_{j,k} \left|\Delta p_j\right| \sqrt{\frac{h_{ik}(\boldsymbol{p}^{(t)}+c\Delta\boldsymbol{p},1)}{p_j^{(t)}+c\Delta p_j}} \sqrt{\frac{h_{ij}(\boldsymbol{p}^{(t)}+c\Delta\boldsymbol{p},1)}{p_k^{(t)}+c\Delta p_k}} \left|\Delta p_k\right|}{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t)} \right\rangle}$$

Since for all $j \in [m]$, $\frac{|\Delta p_j|}{p_j} \leq \frac{1}{4}$, we have that for all $j \in [m]$ and for all $c \in [0, 1]$, $p_j^{(t)} + c\Delta p_j \geq \frac{3}{4} p_j^{(t)}$, which gives:

$$\leq \frac{\epsilon b_i}{2} \frac{\sum_{j,k} \left|\Delta p_j\right| \sqrt{\frac{h_{ik}(\boldsymbol{p}^{(t)}+c\Delta\boldsymbol{p},1)}{\frac{3}{4}p_j^{(t)}}} \sqrt{\frac{h_{ij}(\boldsymbol{p}^{(t)}+c\Delta\boldsymbol{p},1)}{\frac{3}{4}p_k^{(t)}}} \left|\Delta p_k\right|}{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t)} \right\rangle}$$

$$= \frac{2\epsilon b_i}{3} \frac{\sum_{j,k} \left|\Delta p_j\right| \sqrt{\frac{h_{ij}(\boldsymbol{p}^{(t)}+c\Delta\boldsymbol{p},1)}{p_j^{(t)}}} \sqrt{\frac{h_{ik}(\boldsymbol{p}^{(t)}+c\Delta\boldsymbol{p},1)}{p_k^{(t)}}} \left|\Delta p_k\right|}{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t)} \right\rangle}$$

$$= \frac{2\epsilon b_i}{3} \frac{\sum_{j,k\in[m]} \sqrt{\frac{\left|\Delta p_j\right|^2}{p_j^{(t)}} h_{ij}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1) h_{ik}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1) \frac{\left|\Delta p_k\right|^2}{p_k^{(t)}}}}{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t)} \right\rangle}$$

Applying the AM-GM inequality, i.e., for all $x, y \in \mathbb{R}_+$, $\frac{x+y}{2} \geq \sqrt{xy}$, to the sum inside the numerator above, we obtain:

$$\leq \frac{2\epsilon b_i}{3} \frac{\sum_{j,k\in[m]} 1/2 \left(\frac{\left|\Delta p_j\right|^2}{p_j^{(t)}} h_{ij}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1) + h_{ik}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1) \frac{\left|\Delta p_k\right|^2}{p_k^{(t)}} \right)}{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t)} \right\rangle}$$

$$\leq \frac{2\epsilon b_i}{3} \frac{\sum_j \frac{(\Delta p_j)^2}{p_j^{(t)}} h_{ij}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1)}{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t)} \right\rangle}$$

Since for all $j \in [m]$, $\frac{|\Delta p_j|}{p_j^{(t)}} \leq \frac{1}{4}$, we have for all $c \in [0,1]$ that $\frac{4}{5}\sum_j h_{ij}(\boldsymbol{p}^{(t)}, 1)(p_j^{(t)} + c\Delta p_j) \leq \sum_j h_{ij}(\boldsymbol{p}^{(t)}, 1)p_j^{(t)}$:

$$\leq \frac{2\epsilon b_i}{3} \frac{\sum_j \frac{(\Delta p_j)^2}{p_j^{(t)}} h_{ij}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1)}{\frac{4}{5}\sum_j h_{ij}(\boldsymbol{p}^{(t)}, 1)(p_j^{(t)} + c\Delta p_j)}$$

$$= \frac{5\epsilon b_i}{6} \frac{\sum_j \frac{(\Delta p_j)^2}{p_j^{(t)}} h_{ij}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1)}{\sum_j h_{ij}(\boldsymbol{p}^{(t)}, 1)(p_j^{(t)} + c\Delta p_j)}$$

$$\leq \frac{5\epsilon b_i}{6} \frac{\sum_j \frac{(\Delta p_j)^2}{p_j^{(t)}} h_{ij}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1)}{\sum_j h_{ij}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1)(p_j^{(t)} + c\Delta p_j)} \quad \text{(Corollary 7.2.1, Section 7.2)}$$

$$= \frac{5\epsilon}{6} \sum_j \frac{(\Delta p_j)^2}{p_j^{(t)}} d_{ij}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i) \quad \text{(Lemma 7.2.9, Section 7.2)}$$

Because we can bound the change in the expenditure function from one iteration of *tâtonnement* to the next, the Hicksian price elasticity of demand is a better tool with which to analyze the convergence of *tâtonnement* than Marshallian price elasticity of demand. Additionally, as shown previously by Cheung et al. (2013) (Lemma 7.2.4, Section 7.2), we can further upper bound the price terms in Lemma 6.4.2 by the KL divergence between the two prices In light of Theorem 6.1.1, this result suggests that running mirror descent with KL divergence as the Bregman divergence on the dual of the convex program in Theorem 6.2.1 could result in a *tâtonnement* update rule that converges to a Walrasian equilibrium.

6.5 Convergence Bounds for Entropic Tâtonnement

In this section, we analyze the rate of convergence of **entropic tâtonnement**, which corresponds to the *tâtonnement* process given by mirror descent with weighted entropy as the kernel function, i.e., entropic descent. This particular update rule reduces to Equations (6.4) to (6.5), and has been the focus of previous work (Cheung et al., 2013). We provide a sketch of the proof used to obtain our convergence rate in this section. The omitted lemmas and proofs can be found in Appendix 7.2.

At a high level, our proof follows Cheung et al.'s [2013] proof technique for Leontief Fisher markets (Cheung et al., 2013), although we encounter different lower-level technical challenges in generalizing to homothetic markets. This proof technique works as follows. First, we prove that under certain assumptions, the condition required by Theorem 6.1.1 holds when f is the convex potential function for homothetic Fisher markets, i.e., $f = \psi$. For these assumptions to be valid, we need to set γ to be greater than a quadratic function of the maximum absolute value of the price elasticity of the Hicksian demand and the maximum Marshallian demand, for all goods throughout the *tâtonnment* process. Further, since γ needs to be set at the outset, we need to upper bound γ. To do so, we derive a bound on the maximum demand for any good during *tâtonnement* in all homothetic Fisher markets, which in turn allows us to derive an upper bound on γ. Finally, we use Theorem 6.1.1 to obtain the convergence rate of $O(1+\epsilon^2/t)$.

The following lemma derives the conditions under which the antecedent of Theorem 6.1.1 holds for entropic *tâtonnement*.

Lemma 6.5.1.
Consider a homothetic Fisher market $(\boldsymbol{u}, \boldsymbol{b})$ and let $\epsilon = \max_{\boldsymbol{p} \in \Delta_m, j,k \in [m]} \left| \epsilon_{h_{ij}, p_k}(\boldsymbol{p}, 1) \right|$. Then, the following holds for entropic *tâtonnement* when run on $(\boldsymbol{u}, \boldsymbol{b})$: for all $t \in \mathbb{N}$,

$$\psi(\boldsymbol{p}^{(t+1)}) \leq \ell_\psi(\boldsymbol{p}^{(t+1)}, \boldsymbol{p}^{(t)}) + \gamma \mathrm{div}_{\mathrm{KL}}(\boldsymbol{p}^{(t+1)}, \boldsymbol{p}^{(t)}) \ ,$$

where $\gamma = \left(1 + \max_{j \in [m]} \left\{ \sum_{i \in [n]} \max_{t \in \mathbb{N}_+} v_i(\boldsymbol{p}^{(t)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) \right\}\right) \left(6 + \frac{85\epsilon}{12} + \frac{25\epsilon^2}{72}\right)$.

Proof of Lemma 6.5.1

$$
\begin{aligned}
&\psi(\boldsymbol{p}^{(t+1)}) - \ell_\psi(\boldsymbol{p}^{(t+1)}, \boldsymbol{p}^{(t)}) \\
&= \psi(\boldsymbol{p}^{(t+1)}) - \psi(\boldsymbol{p}^{(t)}) + \boldsymbol{z}(\boldsymbol{p}^{(t)}) \cdot \left(\boldsymbol{p}^{(t+1)} - \boldsymbol{p}^{(t)}\right) \\
&= \sum_{j\in[m]}\left(p_j^{(t)} + \Delta p_j\right) - \sum_{i\in[n]} b_i \log\left(e_i(\boldsymbol{p}^{(t+1)}, 1)\right) - \sum_{j\in[m]} p_j^{(t)} + \sum_{i\in[n]} b_i \log\left(e_i(\boldsymbol{p}^{(t)}, 1)\right) + \sum_{j\in[m]} z_j(\boldsymbol{p}^{(t)})\Delta p_j \\
&= \sum_{j\in[m]}\left(p_j^{(t)} + \Delta p_j\right) - \sum_{i\in[n]} b_i \log\left(e_i(\boldsymbol{p}^{(t+1)}, 1)\right) - \sum_{j\in[m]} p_j^{(t)} + \sum_{i\in[n]} b_i \log\left(e_i(\boldsymbol{p}^{(t)}, 1)\right) + \sum_{j\in[m]} (q_j^{(t)} - 1)\Delta p_j \\
&= \sum_{j\in[m]} \Delta p_j q_j^{(t)} - \sum_{i\in[n]} b_i \log\left(e_i(\boldsymbol{p}^{(t+1)}, 1)\right) + \sum_{i\in[n]} b_i \log\left(e_i(\boldsymbol{p}^{(t)}, 1)\right) \\
&= \left\langle \Delta\boldsymbol{p}, \boldsymbol{q}^{(t)} \right\rangle + \sum_{i\in[n]} b_i \log\left(\frac{e_i(\boldsymbol{p}^{(t)}, 1)}{e_i(\boldsymbol{p}^{(t+1)}, 1)}\right) \\
&= \left\langle \Delta\boldsymbol{p}, \boldsymbol{q}^{(t)} \right\rangle + \sum_{i\in[n]} b_i \log\left(\frac{e_i(\boldsymbol{p}^{(t)}, 1)}{e_i(\boldsymbol{p}^{(t)}, 1) + e_i(\boldsymbol{p}^{(t+1)}, 1) - e_i(\boldsymbol{p}^{(t)}, 1)}\right) \\
&= \left\langle \Delta\boldsymbol{p}, \boldsymbol{q}^{(t)} \right\rangle + \sum_{i\in[n]} b_i \log\left(1 - \frac{e_i(\boldsymbol{p}^{(t+1)}, 1) - e_i(\boldsymbol{p}^{(t)}, 1)}{e_i(\boldsymbol{p}^{(t)}, 1)}\left(1 + \frac{e_i(\boldsymbol{p}^{(t+1)}, 1) - e_i(\boldsymbol{p}^{(t)}, 1)}{e_i(\boldsymbol{p}^{(t)}, 1)}\right)^{-1}\right)
\end{aligned}
$$

where the last line is obtained by simply noting that $\forall a, b \in \mathbb{R}, \frac{a}{a+b} = 1 - \frac{b}{a}(1 + \frac{b}{a})^{-1}$. Using Lemma 7.2.12, we then obtain:

$$
\begin{aligned}
&\psi(\boldsymbol{p}^{(t+1)}) - \ell_\psi(\boldsymbol{p}^{(t+1)}, \boldsymbol{p}^{(t)}) \\
&\leq \left\langle \Delta\boldsymbol{p}, \boldsymbol{q}^{(t)} \right\rangle + \sum_{i\in[n]} \left[\left(\frac{4}{3} + \frac{20\epsilon}{27}\right)\sum_{l\in[m]} \frac{d_{il}^{(t)}}{p_l^{(t)}}(\Delta p_l)^2 + \left(\frac{5\epsilon}{6} + \frac{25\epsilon^2}{324}\right)\sum_{l\in[m]} \frac{d_{il}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)}{p_l^{(t)}}(\Delta p_l)^2 - \left\langle \boldsymbol{d}_i^{(t)}, \Delta\boldsymbol{p} \right\rangle\right] \\
&= \left\langle \Delta\boldsymbol{p}, \boldsymbol{q}^{(t)} \right\rangle + \left(\frac{4}{3} + \frac{20\epsilon}{27}\right)\sum_{l\in[m]} \frac{q_l^{(t)}}{p_l^{(t)}}(\Delta p_l)^2 + \left(\frac{5\epsilon}{6} + \frac{25\epsilon^2}{324}\right)\sum_{l\in[m]} \frac{q_l(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, \boldsymbol{b})}{p_l^{(t)}}(\Delta p_l)^2 - \left\langle \boldsymbol{q}^{(t)}, \Delta\boldsymbol{p} \right\rangle \\
&= \left(\frac{4}{3} + \frac{20\epsilon}{27}\right)\sum_{l\in[m]} \frac{q_l^{(t)}}{p_l^{(t)}}(\Delta p_l)^2 + \left(\frac{5\epsilon}{6} + \frac{25\epsilon^2}{324}\right)\sum_{l\in[m]} \frac{q_l(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, \boldsymbol{b})}{p_l^{(t)}}(\Delta p_l)^2 \\
&\leq \left(\frac{4}{3} + \frac{20\epsilon}{27}\right)\sum_{l\in[m]} q_l^{(t)}\left(\frac{9}{2}\right)\mathrm{div}_{\mathrm{KL}}(p_j^{(t)} + \Delta p_j, p_j^{(t)}) + \left(\frac{5\epsilon}{6} + \frac{25\epsilon^2}{324}\right)\sum_{l\in[m]} q_l(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, \boldsymbol{b})\left(\frac{9}{2}\right)\mathrm{div}_{\mathrm{KL}}(p_j^{(t)} + \Delta p_j, p_j^{(t)})
\end{aligned}
$$

where the last line follows from Lemma 7.2.4. Continuing,

$$= \left(6 + \frac{10\epsilon}{3}\right) \sum_{l \in [m]} q_l^{(t)} \mathrm{div}_{\mathrm{KL}}(p_l^{(t)} + \Delta p_l, p_l^{(t)}) + \left(\frac{15\epsilon}{4} + \frac{25\epsilon^2}{72}\right) \sum_{l \in [m]} q_l(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, \boldsymbol{b}) \mathrm{div}_{\mathrm{KL}}(p_l^{(t)} + \Delta p_l, p_l^{(t)})$$

$$\leq \max_{\substack{j \in [m], \\ t \in \mathbb{N}_+}} \left\{q_j^{(t)}\right\} \left(6 + \frac{10\epsilon}{3}\right) \sum_{l \in [m]} \mathrm{div}_{\mathrm{KL}}(p_l^{(t)} + \Delta p_l, p_l^{(t)}) +$$

$$\max_{\substack{j \in [m], \\ t \in \mathbb{N}_+, \\ c \in [0,1]}} \left\{q_j(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p})\right\} \left(\frac{15\epsilon}{4} + \frac{25\epsilon^2}{72}\right) \sum_{l \in [m]} \mathrm{div}_{\mathrm{KL}}(p_l^{(t)} + \Delta p_l, p_l^{(t)})$$

$$\leq \max_{j \in [m], t \in \mathbb{N}_+, c \in [0,1]} \left\{q_j(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p})\right\} \left(6 + \frac{85\epsilon}{12} + \frac{25\epsilon^2}{72}\right) \mathrm{div}_{\mathrm{KL}}(\boldsymbol{p}^{(t)} + \Delta \boldsymbol{p}, \boldsymbol{p}^{(t)}) \tag{6.11}$$

By Lemma 7.2.9, we can re-write the aggregate demand for all goods $j \in [m]$, as follows:

$$\begin{aligned} q_j(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}) &= \sum_{i \in [n]} d_{ij}((1-c)\boldsymbol{p}^{(t)} + c\boldsymbol{p}^{(t+1)}, b_i) \\ &= \sum_{i \in [n]} \frac{h_{ij}((1-c)\boldsymbol{p}^{(t)} + c\boldsymbol{p}^{(t+1)}, 1) b_i}{e_i((1-c)\boldsymbol{p}^{(t)} + c\boldsymbol{p}^{(t+1)}, 1)} \end{aligned}$$

Now, by Danskin's maximum theorem (Danskin, 1966), we know that the expenditure function is concave in prices, that is, for all $c \in [0,1]$, we have $(1-c)e_i(\boldsymbol{p}^{(t)}, 1) + ce_i(\boldsymbol{p}^{(t+1)}, 1) \leq e_i((1-c)\boldsymbol{p}^{(t)} + c\boldsymbol{p}^{(t+1)}, 1)$. Hence, continuing we have for all $j \in [m]$:

$$\begin{aligned} q_j(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}) &= \sum_{i \in [n]} \frac{h_{ij}((1-c)\boldsymbol{p}^{(t)} + c\boldsymbol{p}^{(t+1)}, 1) b_i}{e_i((1-c)\boldsymbol{p}^{(t)} + c\boldsymbol{p}^{(t+1)}, 1)} \\ &\leq \sum_{i \in [n]} \frac{h_{ij}((1-c)\boldsymbol{p}^{(t)} + c\boldsymbol{p}^{(t+1)}, 1) b_i}{(1-c)e_i(\boldsymbol{p}^{(t)}, 1) + ce_i(\boldsymbol{p}^{(t+1)}, 1)} \end{aligned}$$

Further, by Lemma 5 of Goktas et al. (2022), since the expenditure function is homogeneous of degree 0 in prices, notice that we have for all $j \in [m]$, $\max_{\boldsymbol{p} \in \mathbb{R}_+^m / \{\boldsymbol{0}\}} h_{ij}(\boldsymbol{p}, 1) = \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1)$. Note that $\max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1)$ is well-defined since Δ_m is compact, $h_{ij}(\boldsymbol{p}, 1)$ exists for all $\boldsymbol{p} \in \mathbb{R}_+^m$, and is by Berge's maximum theorem (Berge, 1997) continuous in homothetic Fisher markets. Since by the entropic tâtonnement update rule for all time-steps $t \in \mathbb{N}_+$, and goods $j \in [m]$, $p_j^{(t)} > 0$, we then have $h_{ij}((1-c)\boldsymbol{p}^{(t)} + c\boldsymbol{p}^{(t+1)}, 1) \leq \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1)$. Hence, continuing, we have for all $j \in [m]$:

$$q_j(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}) \leq \sum_{i \in [n]} \frac{\max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) b_i}{(1-c)e_i(\boldsymbol{p}^{(t)}, 1) + ce_i(\boldsymbol{p}^{(t+1)}, 1)}$$

Taking a maximum over $c \in [0,1]$ and $t \in \mathbb{N}_+$, and $j \in [m]$, we have for all goods $j \in [m]$:

$$\max_{j\in[m],t\in\mathbb{N}_+,c\in[0,1]} q_j(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}) \leq \max_{j\in[m],t\in\mathbb{N}_+,c\in[0,1]} \sum_{i\in[n]} \frac{\max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p},1)b_i}{(1-c)e_i(\boldsymbol{p}^{(t)},1) + ce_i(\boldsymbol{p}^{(t+1)},1)}$$
$$\leq \max_{j\in[m]} \sum_{i\in[n]} \frac{\max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p},1)b_i}{\min_{t\in\mathbb{N}_+,c\in[0,1]}\{(1-c)e_i(\boldsymbol{p}^{(t)},1) + ce_i(\boldsymbol{p}^{(t+1)},1)\}}$$
$$= \max_{j\in[m]} \sum_{i\in[n]} \frac{\max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p},1)b_i}{\min_{t\in\mathbb{N}_+}\{\min\{e_i(\boldsymbol{p}^{(t)},1), e_i(\boldsymbol{p}^{(t+1)},1)\}\}}$$
$$= \max_{j\in[m]} \sum_{i\in[n]} \frac{\max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p},1)b_i}{\min_{t\in\mathbb{N}_+} e_i(\boldsymbol{p}^{(t)},1)}$$
$$= \max_{j\in[m]} \sum_{i\in[n]} \max_{t\in\mathbb{N}_+} \frac{\max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p},1)b_i}{e_i(\boldsymbol{p}^{(t)},1)}$$
$$= \max_{j\in[m]} \sum_{i\in[n]} \max_{t\in\mathbb{N}_+} v_i(\boldsymbol{p}^{(t)},b_i) \max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p},1)$$

where the last line follows from Corollary 1, Appendix A of Goktas et al. (2022). Plugging the above bound into Equation (6.11), we then obtain the following bound which implies the result:

$$\psi(\boldsymbol{p}^{(t+1)}) - \ell_\psi(\boldsymbol{p}^{(t+1)}, \boldsymbol{p}^{(t)})$$
$$\leq \max_{j\in[m]} \left\{ \sum_{i\in[n]} \max_{t\in\mathbb{N}_+} v_i(\boldsymbol{p}^{(t)},b_i) \max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p},1) \right\} \left(6 + \frac{85\epsilon}{12} + \frac{25\epsilon^2}{72}\right) \mathrm{div}_{\mathrm{KL}}(\boldsymbol{p}^{(t)} + \Delta\boldsymbol{p}, \boldsymbol{p}^{(t)})$$

For the above lemma to be applied in conjuction with Theorem 6.1.1, we have to ensure that the quantity $\max_{j\in[m],t\in\mathbb{N}_+}\left\{\sum_{i\in[n]} v_i(\boldsymbol{p}^{(t)},b_i)\max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p},1)\right\}$ is bounded throughout entropic *tâtonnement* for homothetic Fisher markets. To understand the relevance of this bound, we note that this quantity is an upper bound to the aggregate demand, that is:

$$
\begin{aligned}
q_j^{(t)} &= \sum_{i \in [n]} d_{ij}^{(t)} \\
&= \sum_{i \in [n]} h_{ij}(\boldsymbol{p}^{(t)}, v_i(\boldsymbol{p}^{(t)}, b_i)) \\
&= \sum_{i \in [n]} v_i(\boldsymbol{p}^{(t)}, b_i) h_{ij}(\boldsymbol{p}^{(t)}, 1) && \text{(Lemma 5 of Goktas et al. (2022))} \\
&\leq \sum_{i \in [n]} \min_{t \in \mathbb{N}_+} v_i(\boldsymbol{p}^{(t)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1)
\end{aligned}
$$

As such, proving an upper bound to it implies the excess demand is bounded throughout entropic *tâtonnement*, which in turn implies Lipschitz-smoothness (and hence Bregman-smoothness for any choice of strongly convex kernel function) of the dual of our convex program over all trajectories of entropic *tâtonnement*. The following lemma establishes such a bound and shows that it depends on the initial choice of price $\boldsymbol{p}^{(0)}$, and the maximum possible Hicksian demand to obtain one unit of utility.

Lemma 6.5.2 [Bounded Indirect Utility for Homothetic Fisher Markets]**.**
If entropic *tâtonnement* is run on a homothetic Fisher market $(\boldsymbol{u}, \boldsymbol{b})$, then, for all $t \in \mathbb{N}_+$, the following bound holds:

$$
v_i(\boldsymbol{p}^{(t)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) \leq v_i(\boldsymbol{p}^{(0)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) + 2 \max_{\substack{\boldsymbol{p}, \boldsymbol{q} \in \Delta_m \\ k \in [m] : h_{ik}(\boldsymbol{p}, 1) > 0}} \left\{ \frac{h_{ij}(\boldsymbol{q}, 1)^2}{h_{ik}(\boldsymbol{p}, 1)^2} \right\}
$$

Proof of Lemma 6.5.2

Fix a buyer $i \in [n]$ and good $j \in [m]$. First, note that since by Lemma 5 of Goktas et al. (2022), since the expenditure function is homogeneous of degree 0 in prices, we have for all $j \in [m]$, $\max_{\boldsymbol{p} \in \mathbb{R}_+^m / \{\boldsymbol{0}\}} h_{ij}(\boldsymbol{p}, 1) = \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1)$. In addition, note that $\max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1)$ is well-defined since Δ_m is compact, $h_{ij}(\boldsymbol{p}, 1)$ exists for all $\boldsymbol{p} \in \mathbb{R}_+^m$, and is by Berge's maximum theorem (Berge, 1997) continuous in homothetic Fisher markets. Further, by the entropic tâtonnement update rule for all time-steps $t \in \mathbb{N}_+$, and goods $j \in [m]$, $p_j^{(t)} > 0$, we then have $h_{ij}(\boldsymbol{p}^{(t)}, 1) \leq \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1)$.
We now proceed to prove the claim of the lemma by induction on t.

Base case: $t = 0$. Since $\max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m\\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q},1)^2}{h_{ik}(\boldsymbol{p},1)^2}\right\} \geq 0$, by definition, we have

$$v_i(\boldsymbol{p}^{(0)}, b_i) \max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p},1) \leq v_i(\boldsymbol{p}^{(0)}, b_i) \max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p},1) + 2 \max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m\\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q},1)^2}{h_{ik}(\boldsymbol{p},1)^2}\right\} .$$

Inductive hypothesis. Suppose that for any $t \in \mathbb{N}$, we have:

$$v_i(\boldsymbol{p}^{(t)}, b_i) \max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p},1) \leq v_i(\boldsymbol{p}^{(0)}, b_i) \max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p},1) + 2 \max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m\\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q},1)^2}{h_{ik}(\boldsymbol{p},1)^2}\right\}$$

Inductive step. We will show that the inductive hypothesis holds for $t+1$. We proceed with a proof by cases.

Case 1: $d_{ij}^{(t)} \geq \max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m\\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q},1)}{h_{ik}(\boldsymbol{p},1)}\right\}$.

For all $k \in [m]$, we have:

$$\begin{aligned} d_{ik}^{(t)} &= \frac{h_{ik}^{(t)}}{h_{ij}^{(t)}} d_{ij}^{(t)} && \text{(Lemma 7.2.9, Section 7.2)} \\ &\geq \frac{h_{ik}^{(t)}}{h_{ij}^{(t)}} \max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m\\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q},1)}{h_{ik}(\boldsymbol{p},1)}\right\} \\ &\geq \frac{h_{ik}^{(t)}}{h_{ij}^{(t)}} \frac{h_{ij}^{(t)}}{h_{ik}^{(t)}} = 1 \end{aligned}$$

where the penultimate line follows from the case hypothesis.

The above means that the price of all goods will increase in the next time period, i.e., $\forall k \in [m], p_k^{(t+1)} \geq p_k^{(t)}$ which implies that $e_i(\boldsymbol{p}^{(t+1)}, 1) \geq e_i(\boldsymbol{p}^{(t)}, 1) \geq 0$. In addition, note that the expenditure is positive since prices reach 0 only asymptotically under entropic *tâtonnement*. Which gives us:

$$\begin{aligned} \frac{b_i}{e_i(\boldsymbol{p}^{(t+1)}, 1)} &\leq \frac{b_i}{e_i(\boldsymbol{p}^{(t)}, 1)} \\ v_i(\boldsymbol{p}^{(t+1)}, b_i) &\leq v_i(\boldsymbol{p}^{(t)}, b_i) && \text{(Corollary 1 of Goktas et al. (2022))} \end{aligned}$$

Multiplying both sides by $\max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p},1)$, we have for all $j \in [m]$:

$$\begin{aligned} v_i(\boldsymbol{p}^{(t+1)}, b_i) \max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p},1) &\leq v_i(\boldsymbol{p}^{(t)}, b_i) \max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p},1) \\ &= v_i(\boldsymbol{p}^{(0)}, b_i) \max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p},1) + 2 \max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m\\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q},1)^2}{h_{ik}(\boldsymbol{p},1)^2}\right\} \end{aligned}$$

where the last line follows by the induction hypothesis.

Case 2: $d_{ij}^{(t)} < \max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m \\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q},1)}{h_{ik}(\boldsymbol{p},1)}\right\}$.

For all $k \in [m]$, we have:

$$\begin{aligned}
d_{ik}^{(t)} &= \frac{h_{ik}^{(t)}}{h_{ij}^{(t)}} d_{ij}^{(t)} \\
&\leq \frac{h_{ik}^{(t)}}{h_{ij}^{(t)}} \max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m \\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q},1)}{h_{ik}(\boldsymbol{p},1)}\right\} \\
&= \frac{h_{ik}^{(t)}}{h_{ij}^{(t)}} \frac{h_{ij}^{(t)}}{h_{ik}^{(t)}} = 1
\end{aligned}$$

where the penultimate line follows from the case hypothesis.

The above means that prices of all goods will decrease in the next time period. Now, note that regardless of the aggregate demand $\boldsymbol{q}^{(t)}$ at time $t \in \mathbb{N}$, prices can decrease at most by a factor of $e^{-\frac{1}{5}} \geq 1/2$, that is, for all $j \in [m]$

$$\begin{aligned}
p_j^{(t+1)} &= p_j^{(t)} \exp\left\{\frac{z_j(\boldsymbol{p}^{(t)})}{\gamma}\right\} \\
&= p_j^{(t)} \exp\left\{\frac{q_j^{(t)} - 1}{\gamma}\right\} \\
&\geq p_j^{(t)} \exp\left\{\frac{-1}{\gamma}\right\} \\
&\geq p_j^{(t)} \exp\left\{\frac{-1}{5\max_{\substack{t\in\mathbb{N} \\ j\in[m]}}\{1, q_j^{(t)}\}}\right\} \\
&\geq p_j^{(t)} \exp\left\{\frac{-1}{5}\right\} \\
&\geq p_j^{(t)} e^{-\frac{1}{5}} \geq \frac{1}{2} p_j^{(t)}
\end{aligned}$$

Now, notice that we have $e_i(\boldsymbol{p}^{(t+1)}, 1) \geq e_i(\frac{1}{2}\boldsymbol{p}^{(t)}, 1) = \frac{1}{2} e_i(\boldsymbol{p}^{(t)}, 1) \geq 0$, since the expenditure of the buyer decreases the most when the prices of all goods decrease simultaneously and the expenditure function is homogeneous of degree 1 in prices. In addition, note that the expenditure is positive since prices reach 0 only asymptotically under entropic *tâtonnement*. Hence, we have:

$$\frac{b_i}{e_i(\boldsymbol{p}^{(t+1)}, 1)} \leq 2\frac{b_i}{e_i(\boldsymbol{p}^{(t)}, 1)}$$

$$v_i(\boldsymbol{p}^{(t+1)}, b_i) \leq 2v_i(\boldsymbol{p}^{(t)}, b_i) \qquad \text{(Corollary 1 of Goktas et al. (2022))}$$

Multiplying both sides by $h_{ij}^{(t)}$, and applying Lemma 7.2.9, we have for all $j \in [m]$:

$$v_i(\boldsymbol{p}^{(t+1)}, b_i)h_{ij}^{(t)} \leq 2d_{ij}^{(t)}$$

$$v_i(\boldsymbol{p}^{(t+1)})h_{ij}^{(t)} \leq 2 \max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m \\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q}, 1)}{h_{ik}(\boldsymbol{p}, 1)}\right\} \qquad \text{(Case hypothesis)}$$

Now, taking a minimum over all $j \in [m]$ s.t. $h_{ij}^{(t)} > 0$, we have

$$v_i(\boldsymbol{p}^{(t+1)}) \min_{k\in[m]:h_{ik}^{(t)}>0} h_{ik}^{(t)} \leq 2 \max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m \\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q}, 1)}{h_{ik}(\boldsymbol{p}, 1)}\right\}$$

$$v_i(\boldsymbol{p}^{(t+1)}) \min_{\substack{\boldsymbol{p}\in\Delta_m \\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} h_{ik}(\boldsymbol{p}, 1) \leq 2 \max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m \\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q}, 1)}{h_{ik}(\boldsymbol{p}, 1)}\right\}$$

$$v_i(\boldsymbol{p}^{(t+1)}) \leq \frac{2}{\min_{\substack{\boldsymbol{p}\in\Delta_m \\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} h_{ik}(\boldsymbol{p}, 1)} \max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m \\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q}, 1)}{h_{ik}(\boldsymbol{p}, 1)}\right\}$$

Finally, multiplying both sides by $\max_{\boldsymbol{q}\in\Delta_m} h_{ij}(\boldsymbol{q}, 1)$, we have:

$$\begin{aligned}
v_i(\boldsymbol{p}^{(t+1)}) \max_{\boldsymbol{q}\in\Delta_m} h_{ij}(\boldsymbol{q}, 1) &\leq 2\frac{\max_{\boldsymbol{q}\in\Delta_m} h_{ij}(\boldsymbol{q}, 1)}{\min_{\substack{\boldsymbol{p}\in\Delta_m \\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} h_{ik}(\boldsymbol{p}, 1)} \max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m \\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q}, 1)}{h_{ik}(\boldsymbol{p}, 1)}\right\} \\
&= 2 \left(\max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m \\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q}, 1)}{h_{ik}(\boldsymbol{p}, 1)}\right\}\right) \left(\max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m \\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q}, 1)}{h_{ik}(\boldsymbol{p}, 1)}\right\}\right) \\
&= 2 \max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m \\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q}, 1)}{h_{ik}(\boldsymbol{p}, 1)}\right\}^2 \\
&\leq v_i(\boldsymbol{p}^{(0)}, b_i) \max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p}, 1) + 2 \max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m \\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q}, 1)^2}{h_{ik}(\boldsymbol{p}, 1)^2}\right\}
\end{aligned}$$

Hence, the inductive hypothesis holds for $t + 1$. Putting it all together, we have, for all $t \in \mathbb{N}$:

$$v_i(\boldsymbol{p}^{(t)}, b_i) \max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p}, 1) \leq v_i(\boldsymbol{p}^{(0)}, b_i) \max_{\boldsymbol{p}\in\Delta_m} h_{ij}(\boldsymbol{p}, 1) + 2 \max_{\substack{\boldsymbol{p},\boldsymbol{q}\in\Delta_m \\ k\in[m]:h_{ik}(\boldsymbol{p},1)>0}} \left\{\frac{h_{ij}(\boldsymbol{q}, 1)^2}{h_{ik}(\boldsymbol{p}, 1)^2}\right\}$$

Combining Lemma 6.5.1, and Lemma 6.5.2 with Theorem 6.1.1, we obtain our main result, namely a worst-case convergence rate of $O(^{(1+\epsilon^2)}/_t)$ for entropic *tâtonnement* in homothetic Fisher markets.

Theorem 6.5.1 [Convergence of Entropic Tâtonnement in Homothetic Fisher Markets]**.**
Suppose $(\boldsymbol{u}, \boldsymbol{b})$ is a homothetic Fisher market and $\epsilon = \max_{\boldsymbol{p} \in \Delta_m, j,k \in [m]} \left|\epsilon_{h_{ij}, p_k}(\boldsymbol{p}, 1)\right|$. Then, the following holds for entropic *tâtonnement*: for all $t \in \mathbb{N}$,

$$\psi(\boldsymbol{p}^{(t)}) - \psi(\boldsymbol{p}^*) \leq \frac{\gamma \mathrm{div}_{\mathrm{KL}}(\boldsymbol{p}^*, \boldsymbol{p}^{(0)})}{t} \quad , \tag{6.12}$$

where $\gamma = \left(1 + \max_{j \in [m]} \sum_{i \in [n]} \left[v_i(\boldsymbol{p}^{(0)}, b_i) \max_{\boldsymbol{q} \in \Delta_m} h_{ij}(\boldsymbol{q}, 1) + 2 \max_{\substack{\boldsymbol{p}, \boldsymbol{q} \in \Delta_m \\ k \in [m] : h_{ik}(\boldsymbol{p}, 1) > 0}} \left\{ \frac{h_{ij}(\boldsymbol{q}, 1)^2}{h_{ik}(\boldsymbol{p}, 1)^2} \right\} \right] \right) \left(6 + \frac{85\epsilon}{12} + \frac{25\epsilon^2}{72}\right)$.

Chapter 7

Appendix for Part I

7.1 Details of Section 5.4.3 Experiments

7.1.1 Computational Resources

Our experiments were run on MacOS machine with 8GB RAM and an Apple M1 chip, and took about 10 minutes to run. Only CPU resources were used.

7.1.2 Programming Languages, Packages, and Licensing

We ran our experiments in Python 3.7 (Van Rossum and Drake Jr, 1995), using NumPy (Harris et al., 2020), Jax (Bradbury et al., 2018), and JaxOPT (Blondel et al., 2021). All figures were graphed using Matplotlib (Hunter, 2007).

Python software and documentation are licensed under the PSF License Agreement. Numpy is distributed under a liberal BSD license. Pandas is distributed under a new BSD license. Matplotlib only uses BSD compatible code, and its license is based on the PSF license.

7.1.3 Experimental Setup Details

Each economy is initialized using a random seed to ensure reproducibility. Each consumer is assigned an initial endowment, drawn from a uniform distribution: $\boldsymbol{e}' \sim \mathrm{Unif}(10^{-6}, 1), \quad \forall i \in [n], j \in [m]$. For numerical stability, we restrict the total economy-wide aggregate supply of each commodity to remain fixed at 10[1], to

[1]This is without loss of generality since commodities are divisible.

this end we normalize the endowments of consumers for all $j \in [m]$, $i \in [n]$ to obtain their final endowment:

$$e_{ij} \doteq \frac{10e'_{ij}}{\sum_{i\in[n]} e'_{ij}}.$$

Each consumer's valuation of each commodity is drawn from a uniform distribution, i.e., for all $j \in [m]$, $i \in [n]$:

$$v_{ij} \sim \text{Unif}(0, 1).$$

For any CES consumer $i \in [n]$, the elasticity of substitution parameter ρ_i, is drawn as follows from the uniform distribution for substitutes and complements consumers respectively:

$$\rho_i^{\text{substitutes}} \sim \text{Unif}(0.6, 0.9) \qquad\qquad \rho_i^{\text{complements}} \sim \text{Unif}(-1000, -1)$$

The initial price vector $\boldsymbol{p}^{(0)}$ for the algorithms is drawn from a uniform distribution s.t. for all $j \in [m]$:

$$p_j^{(0)} \sim \text{Unif}(1, 10).$$

We note that while we initialize the prices between 1 and 10 for numerical stability, this choice is without loss of generality since the excess demand is homogeneous of degree 0.

To summarize. Given a random seed, the initialization process consists of: 1) Sampling endowments from a uniform distribution and normalizing them to ensure total supply constraints; 2) sampling valuations from a uniform distribution; 3) sampling substitution parameters for CES consumers, 4) generating an initial price vector.

7.2 Ommited Results and Proofs from Chapter 6

Lemma 7.2.1.

Suppose that u_i is homogeneous, i.e., $\forall \lambda > 0, u_i(\lambda \boldsymbol{x}_i) = \lambda u_i(\boldsymbol{x}_i)$. Then, the expenditure function and the Hicksian demand are homogeneous in ν_i, i.e., for all $\forall \lambda > 0$, $e_i(\boldsymbol{p}, \lambda\nu_i) = \lambda e_i(\boldsymbol{p}, \nu_i)$ and $\boldsymbol{h}_i(\boldsymbol{p}, \lambda\nu_i) = \lambda \boldsymbol{h}_i(\boldsymbol{p}, \nu_i)$. Likewise, the indirect utility function and the Marshallian demand are homogeneous in b_i, i.e., for all $\forall \lambda > 0$, $v_i(\boldsymbol{p}, \lambda b_i) = \lambda v_i(\boldsymbol{p}, b_i)$ and $\boldsymbol{d}_i(\boldsymbol{p}, \lambda b_i) = \lambda \boldsymbol{d}_i(\boldsymbol{p}, b_i)$.

Lemma 7.2.1

Without loss of generality, assume u_i is homogeneous of degree 1.[a]

For Hicksian demand, we have that:

$$\begin{aligned} & \boldsymbol{h}_i(\boldsymbol{p}, \lambda\nu_i) && (7.1)\\ &= \arg\min_{\boldsymbol{x}_i : u_i(\boldsymbol{x}_i) \geq \lambda\nu_i} \boldsymbol{p} \cdot \left(\lambda \frac{\boldsymbol{x}_i}{\lambda}\right) && (7.2)\\ &= \lambda \arg\min_{\boldsymbol{x}_i : u_i(\frac{\boldsymbol{x}_i}{\lambda}) \geq \nu_i} \boldsymbol{p} \cdot \left(\frac{\boldsymbol{x}_i}{\lambda}\right) && (7.3)\\ &= \arg\min_{\boldsymbol{x}_i : u_i(\boldsymbol{x}_i) \geq \nu_i} \boldsymbol{p} \cdot \boldsymbol{x}_i && (7.4)\\ &= \lambda \boldsymbol{h}_i(\boldsymbol{p}, \nu_i) \enspace . && (7.5) \end{aligned}$$

The first equality follows from the definition of Hicksian demand; the second, by the homogeneity of u_i; the third, by the nature of constrained optimization; and the last, from the definition of Hicksian demand again. This result implies homogeneity of the expenditure function in ν_i:

$$e_i(\boldsymbol{p}, \lambda\nu_i) = \boldsymbol{h}_i(\boldsymbol{p}, \lambda\nu_i) \cdot \boldsymbol{p} = \lambda \boldsymbol{h}_i(\boldsymbol{p}, \nu_i) \cdot \boldsymbol{p} = \lambda e_i(\boldsymbol{p}, \nu_i) \enspace .$$

The first and last equalities follow from the definition of the expenditure function, while the second equality follows from the homogeneity of Hicksian demand (Equation (7.5)).

The proof in the case of Marshallian demand and the indirect utility function is analogous.

[a] If the utility function is homogeneous of degree k, we can use a monotonic transformation, namely take the k^{th} root, to transform the utility function into one of degree 1, while still preserving the preferences that it represents.

Lemma 6.2.1.

If u_i is continuous and homogeneous of degree 1, then $v_i(\boldsymbol{p}, b_i)$ and $e_i(\boldsymbol{p}, \nu_i)$ are differentiable in b_i and ν_i, resp. Further, $\mathcal{D}_{b_i} v_i(\boldsymbol{p}, b_i) = \{v_i(\boldsymbol{p}, 1)\}$ and $\mathcal{D}_{\nu_i} e_i(\boldsymbol{p}, \nu_i) = \{e_i(\boldsymbol{p}, 1)\}$.

Lemma 6.2.1

We prove differentiability from first principles:

$$\begin{aligned}
\lim_{h\to 0} \frac{e_i(\boldsymbol{p}, \nu_i + h) - e_i(\boldsymbol{p}, \nu_i)}{h} &= \lim_{h\to 0} \frac{e_i(\boldsymbol{p}, (1)(\nu_i + h)) - e_i(\boldsymbol{p}, (1)\nu_i)}{h} \\
&= \lim_{h\to 0} \frac{e_i(\boldsymbol{p}, 1)(\nu_i + h) - e_i(\boldsymbol{p}, 1)(\nu_i)}{h} \\
&= \lim_{h\to 0} \frac{e_i(\boldsymbol{p}, 1)(\nu_i + h - \nu_i)}{h} \\
&= \lim_{h\to 0} \frac{e_i(\boldsymbol{p}, 1)(h)}{h} \\
&= e_i(\boldsymbol{p}, 1)
\end{aligned}$$

The first line follows from the definition of the derivative; the second line, by homogeneity of the expenditure function (Lemma 7.2.1), since u_i is homogeneous; and the final line follows from the properties of limits. The other two lines follow by simple algebra.

Hence, as $e_i(\boldsymbol{p}, \nu_i)$ is differentiable in ν_i, its subdifferential is a singleton with $\mathcal{D}_{\nu_i} e_i(\boldsymbol{p}, \nu_i) = \{e_i(\boldsymbol{p}, 1)\}$. The proof of the analogous result for the indirect utility function's derivative with respect to b_i is similar .

Corollary 6.2.1.

If buyer i's utility function u_i is CCH, then

$$\frac{1}{e_i(\boldsymbol{p}, 1)} = \frac{1}{\frac{\partial e_i(\boldsymbol{p}, \nu_i)}{\partial \nu_i}} = \frac{\partial v_i(\boldsymbol{p}, b_i)}{\partial b_i} = v_i(\boldsymbol{p}, 1) \quad . \tag{6.6}$$

Proof of Corollary 6.2.1

By Lemma 6.2.1, we know that $e_i(\boldsymbol{p}, \nu_i)$ is differentiable in ν_i and that $\mathcal{D}_{\nu_i} e_i(\boldsymbol{p}, \nu_i) = \{e_i(\boldsymbol{p}, 1)\}$. Similarly, by Lemma 6.2.1, we know that $\mathcal{D}_{b_i} v_i(\boldsymbol{p}, b_i)$ is differentiable in b_i and that $\mathcal{D}_{b_i} v_i(\boldsymbol{p}, b_i) = \{v_i(\boldsymbol{p}, 1)\}$. Combining these facts yields:

$$\begin{aligned}
\mathcal{D}_{\nu_i} e_i(\boldsymbol{p}, \nu_i) \cdot \mathcal{D}_{b_i} v_i(\boldsymbol{p}, b_i) &= e_i(\boldsymbol{p}, 1) \cdot v_i(\boldsymbol{p}, 1) && \text{(Lemma 6.2.1)} \\
&= e_i(\boldsymbol{p}, v_i(\boldsymbol{p}, 1)) && \text{(Lemma 7.2.1)} \\
&= 1 && \text{(Equation (7.12))}
\end{aligned}$$

Therefore, $\frac{1}{\frac{\partial e_i(\boldsymbol{p},\nu_i)}{\partial \nu_i}} = \frac{\partial v_i(\boldsymbol{p},b_i)}{\partial b_i}$. Combining this conclusion with Lemma 6.2.1, we obtain the result.

Lemma 7.2.2.

Given a CCH Fisher market $(\boldsymbol{u}, \boldsymbol{b})$, the dual of our convex program (Theorem 6.2.1) and that of Eisenberg Gale differ by a constant, namely $\sum_{i\in[n]} (b_i \log b_i - b_i)$. In particular,

$$\begin{aligned}
&\min_{\boldsymbol{p}\in\mathbb{R}^m_+} \left\{ \sum_{j\in[m]} p_j - \sum_{i\in[n]} b_i \log\left(\partial_{\nu_i} e_i(\boldsymbol{p},\nu_i)\right) \right\} \\
&= \min_{\boldsymbol{p}\in\mathbb{R}^m_+} \sum_{j\in[m]} p_j + \sum_{i\in[n]} \left(b_i \log\left(v_i(\boldsymbol{p}, b_i)\right) - b_i\right) - \sum_{i\in[n]} (b_i \log b_i - b_i)
\end{aligned}$$

Lemma 7.2.2

$$\begin{aligned}
&\min_{\boldsymbol{p}\in\mathbb{R}^m_+} \sum_{j\in[m]} p_j + \sum_{i\in[n]} \left(b_i \log\left(v_i(\boldsymbol{p}, b_i)\right) - b_i\right) \\
&= \min_{\boldsymbol{p}\in\mathbb{R}^m_+} \sum_{j\in[m]} p_j + \sum_{i\in[n]} b_i \log\left(b_i v_i(\boldsymbol{p}, 1)\right) - \sum_{i\in[n]} b_i && \text{(Lemma 7.2.1)} \\
&= \min_{\boldsymbol{p}\in\mathbb{R}^m_+} \left\{ \sum_{j\in[m]} p_j + \sum_{i\in[n]} b_i \log\left(v_i(\boldsymbol{p}, 1)\right) \right\} + \sum_{i\in[n]} b_i \log b_i - \sum_{i\in[n]} b_i \\
&= \min_{\boldsymbol{p}\in\mathbb{R}^m_+} \left\{ \sum_{j\in[m]} p_j - \sum_{i\in[n]} b_i \log\left(\frac{1}{v_i(\boldsymbol{p}, 1)}\right) \right\} + \sum_{i\in[n]} b_i \log b_i - \sum_{i\in[n]} b_i \\
&= \min_{\boldsymbol{p}\in\mathbb{R}^m_+} \left\{ \sum_{j\in[m]} p_j - \sum_{i\in[n]} b_i \log\left(e_i(\boldsymbol{p}, 1)\right) \right\} + \sum_{i\in[n]} b_i \log b_i - \sum_{i\in[n]} b_i && \text{(Corollary 6.2.1)} \\
&= \min_{\boldsymbol{p}\in\mathbb{R}^m_+} \left\{ \sum_{j\in[m]} p_j - \sum_{i\in[n]} b_i \log\left(\partial_{\nu_i} e_i(\boldsymbol{p},\nu_i)\right) \right\} + \sum_{i\in[n]} b_i \log b_i - \sum_{i\in[n]} b_i && \text{(Lemma 6.2.1)}
\end{aligned}$$

Theorem 6.2.1 [New Convex Program for Homothetic Fisher Markets]**.**

The optimal solutions $(\boldsymbol{X}^*, \boldsymbol{p}^*)$ to the following primal and dual convex programs correspond to Fisher equilibrium allocations and prices, respectively, of the homothetic Fisher market $(\boldsymbol{u}, \boldsymbol{b})$:

Primal

$$\max_{\boldsymbol{X} \in \mathbb{R}_+^{n \times m}} \quad \sum_{i \in [n]} \left[b_i \log u_i \left(\frac{\boldsymbol{x}_i}{b_i} \right) + b_i \right]$$

$$\text{subject to} \quad \sum_{i \in [n]} x_{ij} \leq 1 \qquad \forall j \in [m]$$

Dual

$$\min_{\boldsymbol{p} \in \Delta_m} \psi(\boldsymbol{p}) \doteq \sum_{j \in [m]} p_j - \sum_{i \in [n]} b_i \log \left(e_i(\boldsymbol{p}, 1) \right)$$

Theorem 6.2.1

By Lemma 7.2.2, our dual and the Eisenberg-Gale dual differ by a constant, which is independent of the decision variables $\boldsymbol{p} \in \mathbb{R}_+^m$. Hence, the optimal prices $\boldsymbol{p}^*$ of our dual are the same as those of the Eisenberg-Gale dual, and thus correspond to equilibrium prices in the CCH Fisher market $(\boldsymbol{u}, \boldsymbol{b})$. Finally, the objective function of our convex program's primal is:

$$\sum_{i \in [n]} b_i \log \left(u_i \left(\boldsymbol{x}_i \right) \right) - \sum_{i \in [n]} \left(b_i \log b_i - b_i \right) = \sum_{i \in [n]} b_i \log u_i \left(\frac{\boldsymbol{x}_i}{b_i} \right) + \sum_{i \in [n]} b_i \quad .$$

Danskin's theorem (Danskin, 1966) offers insights into optimization problems of the form: $\min_{\boldsymbol{x} \in X} f(\boldsymbol{x}, \boldsymbol{p})$, where $X \subset \mathbb{R}^m$ is compact and non-empty. Among other things, Danskin's theorem allows us to compute the subdifferential of value of this optimization problem with respect to $\boldsymbol{p}$.

Theorem 7.2.1 [Danskin's Theorem (Danskin, 1966)]**.**

Consider an optimization problem of the form: $\min_{\boldsymbol{x} \in X} f(\boldsymbol{x}, \boldsymbol{p})$, where $X \subset \mathbb{R}^m$ is compact and non-empty. Suppose that X is convex and that f is concave in $\boldsymbol{x}$. Let $V(\boldsymbol{p}) = \min_{\boldsymbol{x} \in X} f(\boldsymbol{x}, \boldsymbol{p})$ and $X^*(\boldsymbol{p}) = \arg\min_{\boldsymbol{x} \in X} f(\boldsymbol{x}, \boldsymbol{p})$. Then the subdifferential of V at $\widehat{\boldsymbol{p}}$ is given by $\mathcal{D}_{\boldsymbol{p}} V(\widehat{\boldsymbol{p}}) = \left\{ \nabla_{\boldsymbol{p}} f(\boldsymbol{x}^*(\widehat{\boldsymbol{p}}), \widehat{\boldsymbol{p}}) \mid \boldsymbol{x}^*(\widehat{\boldsymbol{p}}) \in X^*(\widehat{\boldsymbol{p}}) \right\}$.

Lemma 6.3.1 [Shephard's lemma, generalized for set-valued Hicksian demand (Blume, 2017; Shephard, 2015; Tanaka, 2008)]**.**

Let $e_i(\boldsymbol{p}, \nu_i)$ be the expenditure function of buyer i and $\boldsymbol{h}_i(\boldsymbol{p}, \nu_i)$ be the Hicksian demand set of buyer i. The subdifferential $\mathcal{D}_{\boldsymbol{p}} e_i(\boldsymbol{p}, \nu_i)$ is the Hicksian demand at prices $\boldsymbol{p}$ and utility level ν_i, i.e., $\mathcal{D}_{\boldsymbol{p}} e_i(\boldsymbol{p}, \nu_i) = \boldsymbol{h}_i(\boldsymbol{p}, \nu_i)$.

lemma 6.3.1

Recall that $e_i(\boldsymbol{p}, \nu_i) = \min_{\boldsymbol{x} \in \mathbb{R}^m_+ : u_i(\boldsymbol{x}) \geq \nu_i} \boldsymbol{p} \cdot \boldsymbol{x}$. Without loss of generality, we can assume that consumption set is bounded from above, since utilities are assumed to represent locally non-satiated preferences, i.e., $\min_{\boldsymbol{x} \in X : u_i(\boldsymbol{x}) \geq \nu_i} \boldsymbol{p} \cdot \boldsymbol{x}$ where $X \subset \mathbb{R}^m_+$ is compact. Using Danskin's theorem:

$$\begin{aligned}
\mathcal{D}_{\boldsymbol{p}} e_i(\boldsymbol{p}, \nu_i) &= \left\{ \nabla_{\boldsymbol{p}} \left(\boldsymbol{p} \cdot \boldsymbol{x} \right) \left(\boldsymbol{x}^*(\boldsymbol{p}, \nu_i) \right) \mid \boldsymbol{x}^*(\boldsymbol{p}, \nu_i) \in \boldsymbol{h}_i(\boldsymbol{p}, \nu_i) \right\} && \text{(Danskin's Thm)} \\
&= \left\{ \boldsymbol{x}^*(\boldsymbol{p}, \nu_i) \mid \boldsymbol{x}^*(\boldsymbol{p}, \nu_i) \in \boldsymbol{h}_i(\boldsymbol{p}, \nu_i) \right\} \\
&= \boldsymbol{h}_i(\boldsymbol{p}, \nu_i)
\end{aligned}$$

The first equality follows from Danskin's theorem, using the facts that the objective of the expenditure minimization problem is affine and the constraint set is compact. The second equality follows by calculus, and the third, by the definition of Hicksian demand.

Theorem 6.3.1.

Given any homothetic Fisher market $(\boldsymbol{u}, \boldsymbol{b})$, the subdifferential of the dual of the program in Theorem 6.2.1 at any price $\boldsymbol{p}$ is equal to the negative excess demand in $(\boldsymbol{u}, \boldsymbol{b})$ at price $\boldsymbol{p}$: i.e., $\mathcal{D}_{\boldsymbol{p}} \psi(\boldsymbol{p}) = -\mathcal{Z}(\boldsymbol{p})$.

Theorem 6.3.1

For all goods $j \in [m]$, we have:

$$\begin{aligned}
&\mathcal{D}_{p_j} \left(\sum_{j \in [m]} p_j - \sum_{i \in [n]} b_i \log \partial_{\nu_i} e_i(\boldsymbol{p}, \nu_i) \right) \\
&= \{1\} - \mathcal{D}_{p_j} \left(\sum_{i \in [n]} b_i \log \partial_{\nu_i} e_i(\boldsymbol{p}, \nu_i) \right) \\
&= \{1\} - \sum_{i \in [n]} \mathcal{D}_{p_j} \left(b_i \log \partial_{\nu_i} e_i(\boldsymbol{p}, \nu_i) \right) \\
&= \{1\} - \sum_{i \in [n]} d_{ij}(\boldsymbol{p}, b_i) && \text{(Lemma 6.3.2)} \\
&= -z_j(\boldsymbol{p})
\end{aligned}$$

Lemma 6.3.2.

If buyer i's utility function u_i is continuous and homogeneous, then $\mathcal{D}_{\boldsymbol{p}} \left(b_i \log \left(\frac{\partial e_i(\boldsymbol{p}, \nu_i)}{\partial \nu_i} \right) \right) = \boldsymbol{d}_i(\boldsymbol{p}, b_i)$.

Lemma 6.3.2

Without loss of generality, we can assume u_i is homogeneous of degree 1. Then:

$$\begin{aligned}
\mathcal{D}_{\boldsymbol{p}}\left(b_i \log\left(\frac{\partial e_i(\boldsymbol{p}, \nu_i)}{\partial \nu_i}\right)\right) &= \left(\frac{b_i}{\frac{\partial e_i(\boldsymbol{p}, \nu_i)}{\partial \nu_i}}\right) \mathcal{D}_{\boldsymbol{p}}\left(\frac{\partial e_i(\boldsymbol{p}, \nu_i)}{\partial \nu_i}\right) \\
&= b_i \left(\frac{\partial v_i(\boldsymbol{p}, b_i)}{\partial b_i}\right) \mathcal{D}_{\boldsymbol{p}}\left(\frac{\partial e_i(\boldsymbol{p}, \nu_i)}{\partial \nu_i}\right) && \text{(Corollary 6.2.1)} \\
&= b_i \left(\frac{\partial v_i(\boldsymbol{p}, b_i)}{\partial b_i}\right) \mathcal{D}_{\boldsymbol{p}} e_i(\boldsymbol{p}, 1) && \text{(Lemma 6.2.1)} \\
&= b_i \left(\frac{\partial v_i(\boldsymbol{p}, b_i)}{\partial b_i}\right) \boldsymbol{h}_i(\boldsymbol{p}, 1) && \text{(Shephard's Lemma)} \\
&= b_i\, v_i(\boldsymbol{p}, 1)\, \boldsymbol{h}_i(\boldsymbol{p}, 1) && \text{(Lemma 6.2.1)} \\
&= v_i(\boldsymbol{p}, b_i)\, \boldsymbol{h}_i(\boldsymbol{p}, 1) && \text{(Lemma 7.2.1)} \\
&= \boldsymbol{h}_i\left(\boldsymbol{p}, v_i(\boldsymbol{p}, b_i)\right) && \text{(Lemma 7.2.1)} \\
&= \boldsymbol{d}_i(\boldsymbol{p}, b_i) && \text{(Equation (7.14))}
\end{aligned}$$

We start by presenting the first lemma, which shows that the utility level elasticity of Hicksian demand is equal to 1 in homothetic Fisher markets.

Lemma 6.4.1.

For any Hicksian demand $\boldsymbol{h}_i$ associated with a homogeneous utility function u_i, for all $j, k \in [m], \boldsymbol{p} \in \mathbb{R}^m_+, \nu_i \in \mathbb{R}_+$, it holds that $\epsilon_{h_{ij}, p_k}(\boldsymbol{p}, \nu_i) = \epsilon_{h_{ij}, p_k}(\boldsymbol{p}, 1) = 1$.

Proof of Lemma 6.4.1

Recall from Goktas et al. (2022) that for homogeneous utility functions, the Hicksian demand is homogeneous in ν, i.e., for all $\lambda \geq 0$, $\boldsymbol{h}_i(\boldsymbol{p}, \lambda\nu) = \lambda \boldsymbol{h}_i(\boldsymbol{p}, \nu)$. Hence, we have:

$$\epsilon_{h_{ij},\nu_i}(\boldsymbol{p}, \nu_i) = \mathcal{D}_{\nu_i} h_{ij}(\boldsymbol{p}, \nu_i) \frac{\nu_i}{h_{ij}(\boldsymbol{p}, \nu_i)} \tag{7.6}$$

$$= \nu_i \mathcal{D}_{\nu_i} h_{ij}(\boldsymbol{p}, 1) \frac{\nu_i}{\nu_i h_{ij}(\boldsymbol{p}, 1)} \qquad \text{(Homogeneity of Hicksian demand)} \tag{7.7}$$

$$= \frac{\nu_i}{h_{ij}(\boldsymbol{p}, 1)} \mathcal{D}_{\nu_i} h_{ij}(\boldsymbol{p}, 1) \tag{7.8}$$

$$= \frac{\nu_i}{h_{ij}(\boldsymbol{p}, 1)} \mathcal{D}_{\nu_i} h_{ij}(\boldsymbol{p}, 1) \tag{7.9}$$

$$= \epsilon_{h_{ij},\nu_i}(\boldsymbol{p}, 1) \tag{7.10}$$

Additionally, looking back at Equation (7.9), since Hicksian demand is homogeneous of degree 1 in ν_i for homogeneous utility function (Lemma 7.2.1), by Euler's theorem for homogeneous functions (see, for instance, (Border, 2017)), we have: $\frac{\nu_i}{h_{ij}(\boldsymbol{p},1)} \mathcal{D}_{\nu_i} h_{ij}(\boldsymbol{p}, 1) = \frac{h_{ij}(\boldsymbol{p},1)}{h_{ij}(\boldsymbol{p},1)} = 1$.

We recall Shephard's lemma which was used in the Equation (6.8):

Lemma 6.3.1 [Shephard's lemma, generalized for set-valued Hicksian demand (Blume, 2017; Shephard, 2015; Tanaka, 2008)].

Let $e_i(\boldsymbol{p}, \nu_i)$ be the expenditure function of buyer i and $\boldsymbol{h}_i(\boldsymbol{p}, \nu_i)$ be the Hicksian demand set of buyer i. The subdifferential $\mathcal{D}_{\boldsymbol{p}} e_i(\boldsymbol{p}, \nu_i)$ is the Hicksian demand at prices $\boldsymbol{p}$ and utility level ν_i, i.e., $\mathcal{D}_{\boldsymbol{p}} e_i(\boldsymbol{p}, \nu_i) = \boldsymbol{h}_i(\boldsymbol{p}, \nu_i)$.

We first prove that by setting γ to be 5 times the maximum demand for any good throughout the entropic *tâtonnement* process , we can bound the change in the prices of goods in each round. We will use the fact that the change in the price of each good is bounded as an assumption in most of the following results.

Lemma 7.2.3.

Suppose that entropic tâtonnement process is run for all $t \in [T] \subseteq \mathbb{N}_+$ with $\gamma \geq 5 \max_{\substack{t \in [T] \\ j \in [m]}} \{1, q_j^{(t)}\}$ and let $\Delta \boldsymbol{p} = \boldsymbol{p}^{(t+1)} - \boldsymbol{p}^{(t)}$. then the following holds for all $t \in \mathbb{N}$:

$$e^{-\frac{1}{5}} p_j^{(t)} \leq p_j^{(t+1)} \leq e^{\frac{1}{5}} p_j^{(t)} \text{ and } \frac{|\Delta p_j|}{p_j^{(t)}} \leq \frac{1}{4}$$

Lemma 7.2.3

The price of of a good $j \in [m]$ can at most increase by a factor of $e^{\frac{1}{5}}$:

$$p_j^{(t+1)} = p_j^{(t)} e^{\frac{z_j(\boldsymbol{p}^{(t)})}{\gamma}} = p_j^{(t)} \exp\left\{\frac{q_j^{(t)} - 1}{\gamma}\right\} \leq p_j^{(t)} \exp\left\{\frac{q_j^{(t)}}{\gamma}\right\} \leq p_j^{(t)} \exp\left\{\frac{q_j^{(t)}}{5 \max\limits_{\substack{t \in \mathbb{N} \\ j \in [m]}} \{1, q_j^{(t)}\}}\right\} \leq p_j^{(t)} e^{\frac{1}{5}}$$

and decrease by a factor of $e^{-\frac{1}{5}}$:

$$p_j^{(t+1)} = p_j^{(t)} e^{\frac{z_j(\boldsymbol{p}^{(t)})}{\gamma}} = p_j^{(t)} \exp\left\{\frac{q_j^{(t)} - 1}{\gamma}\right\} \geq p_j^{(t)} \exp\left\{\frac{-1}{\gamma}\right\} \geq p_j^{(t)} \exp\left\{\frac{-1}{5 \max\limits_{\substack{t \in \mathbb{N} \\ j \in [m]}} \{1, q_j^{(t)}\}}\right\} \geq p_j^{(t)} e^{-\frac{1}{5}}$$

Hence, we have $e^{-\frac{1}{5}} p_j^{(t)} \leq p_j^{(t+1)} \leq e^{\frac{1}{5}} p_j^{(t)}$. Substracting $p_j^{(t)}$ from both sides and dividing by $p_j^{(t)}$, we obtain:

$$\frac{|\Delta p_j|}{p_j^{(t)}} = \frac{|p_j^{(t+1)} - p_j^{(t)}|}{p_j^{(t)}} \leq e^{1/5} - 1 \leq \frac{1}{4}$$

The following two results are due to Cheung et al. (2013). We include their proofs for completeness. They allows us to relate the change in prices to the KL-divergence.

Lemma 7.2.4 [Cheung et al. (2013)]**.**

Fix $t \in \mathbb{N}_+$ and let $\Delta \boldsymbol{p} = \boldsymbol{p}^{(t+1)} - \boldsymbol{p}^{(t)}$. Suppose that for all $j \in [m]$, $\frac{|\Delta p_j|}{p_j^{(t)}} \leq \frac{1}{4}$, then:

$$\frac{(\Delta p_j)^2}{p_j^{(t)}} \leq \frac{9}{2} \mathrm{div}_{\mathrm{KL}}(p_j^{(t)} + \Delta p_j, p_j^{(t)}) \tag{7.11}$$

Lemma 7.2.4

The bound $\log(x) \geq x - x^2$ for $|x|\leq \frac{1}{4}$ is used below:

$$\begin{aligned}
\mathrm{div}_{\mathrm{KL}}(p_j^{(t)} + \Delta p_j, p_j^{(t)}) &= (p_j^{(t)} + \Delta p_j)(\log(p_j^{(t)} + \Delta p_j)) - (p_j^{(t)} + \Delta p_j - p_j^{(t)} \log(p_j) + p_j^{(t)} - \log(p_j^{(t)})\Delta p_j \\
&= -\Delta p_j + (p_j^{(t)} + \Delta p_j) \log\left(1 + \frac{\Delta p_j}{p_j^{(t)}}\right) \\
&\geq -\Delta p_j + (p_j^{(t)} + \Delta p_j)\left(\frac{\Delta p_j}{p_j^{(t)}} - \frac{11}{18}\frac{(\Delta p_j)^2}{(p_j^{(t)})^2}\right) \\
&\geq \frac{7}{18}\frac{(\Delta p_j)^2}{p_j^{(t)}}\left(1 - \frac{11}{7}\frac{\Delta p_j}{p_j^{(t)}}\right) \\
&= \frac{7}{18}\frac{17}{28}\frac{(\Delta p_j)^2}{p_j^{(t)}} \\
&\geq \frac{2}{9}\frac{(\Delta p_j)^2}{p_j^{(t)}}
\end{aligned}$$

Lemma 7.2.5.

Fix $t \in \mathbb{N}_+$ and let $\Delta \boldsymbol{p} = \boldsymbol{p}^{(t+1)} - \boldsymbol{p}^{(t)}$. Suppose that $\frac{|\Delta p_j|}{p_j} \leq \frac{1}{4}$, then for any $c \in (0, 1)$, and $\boldsymbol{A} \in \mathbb{R}^{n \times m}$, and for all $j \in [m]$:

$$\frac{1}{b_i}\sum_{j\in[m]}\sum_{k\in[m]} a_{il} d_{ik}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)|\Delta p_j||\Delta p_k|\leq \frac{4}{3}\sum_{l\in[m]}\frac{a_{il}}{p_l^{(t)}}(\Delta p_l)^2$$

Lemma 7.2.5

First, note that since by our assumption the utilities are locally non-satiated, Walras' law is satisfied, i.e., we have $b_i = \sum_{k\in[m]} d_{ik}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)(p_k^{(t)} + c\Delta p_k)$;

$$
\begin{aligned}
& b_i \sum_{l\in[m]} \frac{a_{il}}{p_l^{(t)}}(\Delta p_l)^2 \\
&= \sum_{l\in[m]} \frac{\left(\sum_{k\in[m]} d_{ik}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)(p_k^{(t)} + c\Delta p_k)\right) d_{il}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)}{p_l^{(t)}}(\Delta p_l)^2 \\
&\geq \sum_{l\in[m]} \frac{\left(\sum_{k\in[m]} d_{ik}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)(p_k^{(t)} - \frac{1}{4}p_k^{(t)})\right) a_{il}}{p_l^{(t)}}(\Delta p_l)^2 \\
&= \sum_{l\in[m]} \frac{\left(\sum_{k\in[m]} d_{ik}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)(\frac{3}{4}p_k^{(t)})\right) a_{il}}{p_l^{(t)}}(\Delta p_l)^2 \\
&= \frac{3}{4}\sum_{l\in[m]}\sum_{k\in[m]} a_{il} d_{il}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)\frac{p_k^{(t)}}{p_l^{(t)}}(\Delta p_l)^2 \\
&= \frac{3}{4}\left[\sum_{l\in[m]} a_{il} d_{il}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)(\Delta p_l)^2 + \sum_{l\in[m]}\sum_{k\neq l} a_{il} d_{ik}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)\frac{p_k^{(t)}}{p_l^{(t)}}(\Delta p_l)^2\right] \\
&= \frac{3}{4}\left[\sum_{l\in[m]} a_{il} d_{il}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)(\Delta p_l)^2 + \sum_{k\in[m]}\sum_{k\leq l} a_{il} d_{ik}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)\left(\frac{p_k^{(t)}}{p_l^{(t)}}|\Delta p_l|^2 + \frac{p_l^{(t)}}{p_k^{(t)}}|\Delta p_k|^2\right)\right]
\end{aligned}
$$

Now, we apply the AM-GM inequality, i.e., for all $x, y \in \mathbb{R}_+$ since $\sqrt{xy} \leq \frac{x+y}{2}$, we have:

$$
\begin{aligned}
b_i \sum_{l\in[m]} \frac{a_{il}}{p_l^{(t)}}(\Delta p_l)^2 &\geq \frac{3}{4}\sum_{l\in[m]} a_{il} d_{il}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)(\Delta p_l)^2 + \sum_{k<l} a_{il} d_{ik}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)\left(2|\Delta p_l||\Delta p_k|\right) \\
&= \frac{3}{4}\sum_{j\in[m]}\sum_{k\in[m]} a_{il} d_{ik}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)|\Delta p_j||\Delta p_k|
\end{aligned}
$$

Lemma 7.2.6.

(Cheung et al., 2013) For all $j \in [m]$:

$$
\frac{1}{b_i}\sum_{j\in[m]}\sum_{k\in[m]} d_{ij}^{(t)} d_{ik}^{(t)}|\Delta p_j||\Delta p_k| \leq \sum_{l\in[m]} \frac{d_{il}^{(t)}}{p_l^{(t)}}(\Delta p_l)^2
$$

Lemma 7.2.6

First, note that by Walras' law we have $b_i = \sum_{k\in[m]} d_{ik}^{(t)} p_k^{(t)}$;

$$
\begin{aligned}
b_i \sum_{l\in[m]} \frac{d_{il}^{(t)}}{p_l^{(t)}}(\Delta p_l)^2 &= \sum_{l\in[m]} \frac{\left(\sum_{k\in[m]} d_{ik}^{(t)} p_k^{(t)}\right) d_{il}^{(t)}}{p_l^{(t)}}(\Delta p_l)^2 \\
&= \sum_{l\in[m]} \sum_{k\in[m]} d_{il}^{(t)} d_{ik}^{(t)} \frac{p_k^{(t)}}{p_l^{(t)}}(\Delta p_l)^2 \\
&= \sum_{l\in[m]} (d_{il}^{(t)})^2(\Delta p_l)^2 + \sum_{l\in[m]} \sum_{k\neq l} d_{il}^{(t)} d_{ik}^{(t)} \frac{p_k^{(t)}}{p_l^{(t)}}(\Delta p_l)^2 \\
&= \sum_{l\in[m]} (d_{il}^{(t)})^2(\Delta p_l)^2 + \sum_{k\in[m]} \sum_{k\leq l} d_{ik}^{(t)} d_{il}^{(t)} \left(\frac{p_k^{(t)}}{p_l^{(t)}}(\Delta p_l)^2 + \frac{p_l^{(t)}}{p_k^{(t)}}(\Delta p_k)^2\right)
\end{aligned}
$$

Now, we apply the AM-GM inequality:

$$
\begin{aligned}
b_i \sum_{l\in[m]} \frac{d_{il}^{(t)}}{p_l^{(t)}}(\Delta p_l)^2 &\geq \sum_{l\in[m]} (d_{il}^{(t)})^2(\Delta p_l)^2 + \sum_{k<l} d_{ik}^{(t)} d_{il}^{(t)} \left(2|\Delta p_l||\Delta p_k|\right) \\
&= \sum_{j\in[m]} \sum_{k\in[m]} d_{ij}^{(t)} d_{ik}^{(t)} |\Delta p_j||\Delta p_k|
\end{aligned}
$$

An important result in microeconomics is the **law of demand** which states that when the price of a good increases, the Hicksian demand for that good decreases in a very general setting of utility functions (Levin, 2004; Mas-Colell et al., 1995). We state a weaker version of the law of demand which is re-formulated to fit the tâtonnement framework.

Lemma 7.2.7 [Law of Demand]**.**
(Levin, 2004; Mas-Colell et al., 1995) Suppose that $\forall j \in [m], t \in \mathbb{N}, p_j^{(t)}, p_j^{(t+1)} \geq 0$. Then, $\sum_{j\in[m]} \Delta p_j \left(h_{ij}^{(t+1)} - h_{ij}^{(t)}\right) \leq 0$.

A simple corollary of the law of demand which is used throughout the rest of this paper is that, during tâtonnement, the change in expenditure of the next time period is always less than or equal to the change in expenditure of the previous time period's.

Corollary 7.2.1.
Suppose that $\forall t \in \mathbb{N}, j \in [m], p_j^{(t)}, p_j^{(t+1)} \geq 0$, then $\forall t \in \mathbb{N}, \sum_{j\in[m]} \Delta p_j h_{ij}^{(t+1)} \leq \sum_{j\in[m]} \Delta p_j h_{ij}^{(t)}$.

The following lemma simply restates an essential fact about expenditure functions and Hicksian demand, namely that the Hicksian demand is the minimizer of the expenditure function.

Lemma 7.2.8.

For all $t \in \mathbb{N}$, we have $\sum_{j\in[m]} h_{ij}^{(t)} p_j^{(t)} \leq \sum_{j\in[m]} h_{ij}^{(t+1)} p_j^{(t)}$.

Lemma 7.2.8

For the sake of contradiction, assume that $\sum_{j\in[m]} h_{ij}^{(t)} p_j^{(t)} > \sum_{j\in[m]} h_{ij}^{(t+1)} p_j^{(t)}$. By the definition of the Hicksian demand, we know that the bundle $\boldsymbol{h}_i^{(t)}$ provides the buyer with one unit of utility. Recall that the expenditure at any price $\boldsymbol{p}$ is equal to the sum of the product of the Hicksian demands and prices, that is $e_i(\boldsymbol{p}, 1) = \sum_{j\in[m]} h_{ij}(\boldsymbol{p}, 1) p_j$. Hence, we have $e_i(p_j^{(t)}, 1) = \sum_{j\in[m]} h_{ij}^{(t)} p_j^{(t)} > \sum_{j\in[m]} h_{ij}^{(t+1)} p_j^{(t)} = e_i(\boldsymbol{p}^{(t)}, 1)$, a contradiction.

We now introduce the following lemma which makes use of results on the behavior of Hicksian demand and expenditure functions in homothetic Fisher markets introduced by Goktas et al. (2022). In conjunction with Corollary 7.2.1 and Lemma 7.2.8 are key in proving that Lemma 6.5.1 holds allowing us to establish convergence of tâtonnement in a general setting of utility functions. Additionally, the lemma relates the Marshallian demand of homogeneous utility functions to their Hicksian demand. Before we present the lemma, we recall the following identities (Mas-Colell et al., 1995):

$$\forall b_i \in \mathbb{R}_+ \qquad e_i(\boldsymbol{p}, v_i(\boldsymbol{p}, b_i)) = b_i \tag{7.12}$$

$$\forall \nu_i \in \mathbb{R}_+ \qquad v_i(\boldsymbol{p}, e_i(\boldsymbol{p}, \nu_i)) = \nu_i \tag{7.13}$$

$$\forall b_i \in \mathbb{R}_+ \qquad \boldsymbol{h}_i(\boldsymbol{p}, v_i(\boldsymbol{p}, b_i)) = \boldsymbol{d}_i(\boldsymbol{p}, b_i) \tag{7.14}$$

$$\forall \nu_i \in \mathbb{R}_+ \qquad \boldsymbol{d}_i(\boldsymbol{p}, e_i(\boldsymbol{p}, \nu_i)) = \boldsymbol{h}_i(\boldsymbol{p}, \nu_i) \tag{7.15}$$

Lemma 7.2.9.

Suppose that u_i is continuous and homogeneous, then the following holds:

$$\forall j \in [m] \qquad d_{ij}(\boldsymbol{p}, b_i) = \frac{b_i h_{ij}(\boldsymbol{p}, 1)}{\sum_{j\in[m]} h_{ij}(\boldsymbol{p}, 1) p_j}$$

Lemma 7.2.9

We note that when utility function u_i is strictly concave, the Marshallian and Hicksian demand are unique making the following equalities well-defined.

$$\begin{aligned}
\frac{b_i h_{ij}(\boldsymbol{p}, 1)}{\sum_{j \in [m]} h_{ij}(\boldsymbol{p}, 1) p_j} &= \frac{b_i h_{ij}(\boldsymbol{p}, 1)}{e_i(\boldsymbol{p}, 1)} && \text{(Definition of expenditure function)} \\
&= b_i v_i(\boldsymbol{p}, 1) h_{ij}(\boldsymbol{p}, 1) && \text{(Corollary 1 of Goktas et al. (2022))} \\
&= v_i(\boldsymbol{p}, b_i) h_{ij}(\boldsymbol{p}, 1) \\
&= h_{ij}(\boldsymbol{p}, v_i(\boldsymbol{p}, b_i)) \\
&= d_{ij}(\boldsymbol{p}, b_i) && \text{(Marshallian Demand Identity Equation (7.14))}
\end{aligned}$$

The following lemma proves that the relative change in expenditures at each iteration of tatonnement is bounded when the relative change in prices is bounded.

Lemma 7.2.10.

Suppose that $\forall j \in [m]$, $\frac{|\Delta p_j|}{p_j^{(t)}} \leq \frac{1}{4}$, then for any $t \in \mathbb{N}_+$ and $i \in [n]$:

$$\left| \frac{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t+1)}, 1), \boldsymbol{p}^{(t+1)} \right\rangle - \left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t)} \right\rangle}{\left\langle \boldsymbol{h}_i^{(t)}, \boldsymbol{p}^{(t)} \right\rangle} \right| \leq \frac{1}{4} \tag{7.16}$$

Proof

Case 1: $\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t+1)}, 1), \boldsymbol{p}^{(t+1)} \right\rangle \geq \left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t)} \right\rangle$

$$\frac{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t+1)}, 1), \boldsymbol{p}^{(t+1)} \right\rangle - \left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t)} \right\rangle}{\left\langle \boldsymbol{h}_i^{(t)}, \boldsymbol{p}^{(t)} \right\rangle} \tag{7.17}$$

$$\leq \frac{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t+1)} \right\rangle - \left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t)} \right\rangle}{\left\langle \boldsymbol{h}_i^{(t)}, \boldsymbol{p}^{(t)} \right\rangle} \quad \text{(Corollary 7.2.1)} \tag{7.18}$$

$$= \frac{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t+1)} \right\rangle}{\left\langle \boldsymbol{h}_i^{(t)}, \boldsymbol{p}^{(t)} \right\rangle} - 1 \tag{7.19}$$

$$\leq \frac{5}{4} \frac{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t)} \right\rangle}{\left\langle \boldsymbol{h}_i^{(t)}, \boldsymbol{p}^{(t)} \right\rangle} - 1 \tag{7.20}$$

$$= \frac{1}{4} \tag{7.21}$$

where the penultimate line follows from the assumption that $\forall j \in [m], \frac{|\Delta p_j|}{p_j^{(t)}} \leq \frac{1}{4}$.

Case 2: $\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t+1)}, 1), \boldsymbol{p}^{(t+1)} \right\rangle \leq \left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t)} \right\rangle$

$$\frac{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t)} \right\rangle - \left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t+1)}, 1), \boldsymbol{p}^{(t+1)} \right\rangle}{\left\langle \boldsymbol{h}_i^{(t)}, \boldsymbol{p}^{(t)} \right\rangle} = 1 - \frac{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t+1)}, 1), \boldsymbol{p}^{(t+1)} \right\rangle}{\left\langle \boldsymbol{h}_i^{(t)}, \boldsymbol{p}^{(t)} \right\rangle} \tag{7.22}$$

$$\leq 1 - \frac{3}{4} \frac{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t+1)}, 1), \boldsymbol{p}^{(t)} \right\rangle}{\left\langle \boldsymbol{h}_i^{(t)}, \boldsymbol{p}^{(t)} \right\rangle} \tag{7.23}$$

$$\leq 1 - \frac{3}{4} \frac{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \boldsymbol{p}^{(t)} \right\rangle}{\left\langle \boldsymbol{h}_i^{(t)}, \boldsymbol{p}^{(t)} \right\rangle} \qquad \text{(Corollary 7.2.1)} \tag{7.24}$$

$$= \frac{1}{4} \tag{7.25}$$

where the second line follows from the assumption that $\forall j \in [m], \frac{|\Delta p_j|}{p_j^{(t)}} \leq \frac{1}{4}$.

Lemma 7.2.11.

Suppose that for all $j \in [m]$, $\frac{|\Delta p_j|}{p_j} \leq \frac{1}{4}$, then for some $c \in (0, 1)$ and $t \in \mathbb{N}_+$, we have:

$$\frac{1}{b_i} \left(\left\langle \boldsymbol{d}_i^{(t)}, \Delta \boldsymbol{p} \right\rangle + \frac{b_i}{2} \frac{\left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)} \right)^2 \tag{7.26}$$

$$\leq \left(1 + \frac{5\epsilon}{9}\right) \sum_{l \in [m]} \frac{d_{il}^{(t)}}{p_l^{(t)}} (\Delta p_l)^2 + \left(\frac{25\epsilon^2}{432}\right) \sum_{l \in [m]} \frac{d_{il}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i)}{p_l^{(t)}} (\Delta p_l)^2 \tag{7.27}$$

Proof of Lemma 7.2.11

$$\frac{1}{b_i}\left(\left\langle \boldsymbol{d}_i^{(t)}, \Delta\boldsymbol{p}\right\rangle + \frac{b_i}{2}\frac{\left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1)\Delta\boldsymbol{p}, \Delta\boldsymbol{p}\right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)}\right)^2$$

$$= \frac{1}{b_i}\left[\left\langle \boldsymbol{d}_i^{(t)}, \Delta\boldsymbol{p}\right\rangle^2 + 2\left\langle \boldsymbol{d}_i^{(t)}, \Delta\boldsymbol{p}\right\rangle\left(\frac{b_i}{2}\frac{\left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1)\Delta\boldsymbol{p}, \Delta\boldsymbol{p}\right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)}\right) + \left(\frac{b_i}{2}\frac{\left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1)\Delta\boldsymbol{p}, \Delta\boldsymbol{p}\right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)}\right)^2\right]$$

$$\leq \frac{1}{b_i}\left[\left|\left\langle \boldsymbol{d}_i^{(t)}, \Delta\boldsymbol{p}\right\rangle^2\right| + 2\left|\left\langle \boldsymbol{d}_i^{(t)}, \Delta\boldsymbol{p}\right\rangle\right|\left|\frac{b_i}{2}\frac{\left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1)\Delta\boldsymbol{p}, \Delta\boldsymbol{p}\right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)}\right| + \left|\frac{b_i}{2}\frac{\left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1)\Delta\boldsymbol{p}, \Delta\boldsymbol{p}\right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)}\right|^2\right]$$

$$\leq \frac{1}{b_i}\left[\left\langle \boldsymbol{d}_i^{(t)}, |\Delta\boldsymbol{p}|\right\rangle^2 + 2\left\langle \boldsymbol{d}_i^{(t)}, |\Delta\boldsymbol{p}|\right\rangle\left|\frac{b_i}{2}\frac{\left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1)\Delta\boldsymbol{p}, \Delta\boldsymbol{p}\right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)}\right| + + \left|\frac{b_i}{2}\frac{\left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, 1)\Delta\boldsymbol{p}, \Delta\boldsymbol{p}\right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)}\right|^2\right]$$

where we denote $|\Delta\boldsymbol{p}| = (|\Delta p_1|, \ldots, |\Delta p_m|)$.

$$\leq \frac{1}{b_i}\left[\left\langle \boldsymbol{d}_i^{(t)}, |\Delta\boldsymbol{p}|\right\rangle^2 + 2\left\langle \boldsymbol{d}_i^{(t)}, |\Delta\boldsymbol{p}|\right\rangle \left(\frac{5\epsilon}{6}\sum_j \frac{(\Delta p_j)^2}{p_j} d_{ij}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)\right) + \left(\frac{5\epsilon}{6}\sum_j \frac{(\Delta p_j)^2}{p_j} d_{ij}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)\right)^2\right] \quad \text{(Lemma 6.4.2)}$$

$$= \frac{1}{b_i}\left[\left\langle \boldsymbol{d}_i^{(t)}, |\Delta\boldsymbol{p}|\right\rangle^2 + \frac{5\epsilon}{3}\left\langle \boldsymbol{d}_i^{(t)}, |\Delta\boldsymbol{p}|\right\rangle\left(\sum_j \frac{(\Delta p_j)^2}{p_j} d_{ij}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)\right) + \frac{25\epsilon^2}{36}\left(\sum_j \frac{(\Delta p_j)^2}{p_j} d_{ij}(\boldsymbol{p}^{(t)} + c\Delta\boldsymbol{p}, b_i)\right)^2\right]$$

Since $\forall j \in [m]$, $\frac{|\Delta p_j|}{p_j^{(t)}} \leq \frac{1}{4}$, we have:

$$\leq \frac{1}{b_i}\left[\left\langle \boldsymbol{d}_i^{(t)}, |\Delta \boldsymbol{p}|\right\rangle^2 + \frac{5\epsilon}{12}\left\langle \boldsymbol{d}_i^{(t)}, |\Delta \boldsymbol{p}|\right\rangle \left(\sum_j \left|\Delta p_j\right| d_{ij}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1)\right)\right.$$
$$\left. + + \frac{25\epsilon^2}{576}\left(\sum_j |\Delta p_j| d_{ij}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i)\right)^2\right] \tag{7.28}$$
$$= \frac{1}{b_i}\sum_{j\in[m]}\sum_{k\in[m]} d_{ij}^{(t)} d_{ik}^{(t)} |\Delta p_j||\Delta p_k| + \frac{1}{b_i}\frac{5\epsilon}{12}\sum_j\sum_k d_{ik}^{(t)} d_{ij}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i)\left|\Delta p_k\right|\left|\Delta p_j\right|$$
$$+ \frac{1}{b_i}\frac{25\epsilon^2}{576}\sum_j\sum_k d_{ij}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i) d_{ik}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i)|\Delta p_k||\Delta p_j| \tag{7.29}$$
$$\leq \sum_{l\in[m]} \frac{d_{il}^{(t)}}{p_l^{(t)}}(\Delta p_l)^2 + \frac{1}{b_i}\frac{5\epsilon}{12}\sum_j\sum_k d_{ik}^{(t)} d_{ij}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i)\left|\Delta p_k\right|\left|\Delta p_j\right|$$
$$+ \frac{1}{b_i}\frac{25\epsilon^2}{576}\sum_j\sum_k d_{ij}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i) d_{ik}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i)|\Delta p_k||\Delta p_j| \tag{7.30}$$

where the last line was obtained by (Lemma 7.2.6). Continuing, by Lemma 7.2.5, we have:

$$\leq \sum_{l\in[m]} \frac{d_{il}^{(t)}}{p_l^{(t)}}(\Delta p_l)^2 + \frac{5\epsilon}{12}\frac{4}{3}\sum_{l\in[m]} \frac{d_{il}^{(t)}}{p_l^{(t)}}(\Delta p_l)^2 + \frac{25\epsilon^2}{576}\frac{4}{3}\sum_{l\in[m]} \frac{d_{il}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i)}{p_l^{(t)}}(\Delta p_l)^2 \tag{7.31}$$
$$= \left(1 + \frac{5\epsilon}{9}\right)\sum_{l\in[m]} \frac{d_{il}^{(t)}}{p_l^{(t)}}(\Delta p_l)^2 + \left(\frac{25\epsilon^2}{432}\right)\sum_{l\in[m]} \frac{d_{il}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i)}{p_l^{(t)}}(\Delta p_l)^2 \tag{7.32}$$

Lemma 7.2.12.

Suppose that $\frac{|\Delta p_j|}{p_j^{(t)}} \leq \frac{1}{4}$, then

$$b_i \log\left(1 - \frac{e_i(\boldsymbol{p}^{(t+1)}, 1) - e_i(\boldsymbol{p}^{(t)}, 1)}{e_i(\boldsymbol{p}^{(t)}, 1)}\left(1 + \frac{e_i(\boldsymbol{p}^{(t+1)}, 1) - e_i(\boldsymbol{p}^{(t)}, 1)}{e_i(\boldsymbol{p}^{(t)}, 1)}\right)^{-1}\right)$$
$$\leq \left(\frac{4}{3} + \frac{20\epsilon}{27}\right)\sum_{l\in[m]} \frac{d_{il}^{(t)}}{p_l^{(t)}}(\Delta p_l)^2 + \left(\frac{5\epsilon}{6} + \frac{25\epsilon^2}{324}\right)\sum_{l\in[m]} \frac{(\Delta p_l)^2}{p_l^{(t)}} d_{il}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i) - \left\langle \boldsymbol{d}_i(\boldsymbol{p}^{(t)}, 1), \Delta \boldsymbol{p}\right\rangle$$

Proof of Lemma 7.2.12

First, we note that $\boldsymbol{h}_i^{(t)} \cdot \boldsymbol{p}^{(t)} > 0$ because prices during our tâtonnement rule reach 0 only asymptotically and Hicksian demand for one unit of utility at prices $\boldsymbol{p}^{(t)} > 0$ is strictly positive; and likewise, prices reach ∞ only asymptotically, which implies that Hicksian demand is always strictly positive. This fact will come handy, as we divide some expressions by $\boldsymbol{h}_i^{(t)} \cdot \boldsymbol{p}^{(t)}$.

Fix $t \in \mathbb{N}_+$ and $i \in [n]$. Since by our assumptions $\frac{|\Delta p_j|}{p_j^{(t)}} \leq \frac{1}{4}$, by Lemma 7.2.10, we have $0 \leq \left| \frac{e_i(\boldsymbol{p}^{(t+1)},1) - e_i(\boldsymbol{p}^{(t)},1)}{e_i(\boldsymbol{p}^{(t)},1)} \right| \leq \frac{1}{4}$. We can then use the bound $1 - x(1+x)^{-1} \leq 1 + \frac{4}{3}x^2 - x$, for $0 \leq |x| \leq \frac{1}{4}$, with $x = \frac{e_i(\boldsymbol{p}^{(t+1)},1) - e_i(\boldsymbol{p}^{(t)},1)}{e_i(\boldsymbol{p}^{(t)},1)}$, to get:

$$
\begin{aligned}
& b_i \log \left(1 - \frac{e_i(\boldsymbol{p}^{(t+1)},1) - e_i(\boldsymbol{p}^{(t)},1)}{e_i(\boldsymbol{p}^{(t)},1)} \left(1 + \frac{e_i(\boldsymbol{p}^{(t+1)},1) - e_i(\boldsymbol{p}^{(t)},1)}{e_i(\boldsymbol{p}^{(t)},1)} \right)^{-1} \right) \\
& \leq b_i \log \left(1 + \frac{4}{3} \left(\frac{e_i(\boldsymbol{p}^{(t+1)},1) - e_i(\boldsymbol{p}^{(t)},1)}{e_i(\boldsymbol{p}^{(t)},1)} \right)^2 - \frac{e_i(\boldsymbol{p}^{(t+1)},1) - e_i(\boldsymbol{p}^{(t)},1)}{e_i(\boldsymbol{p}^{(t)},1)} \right)
\end{aligned}
$$

Let $a = \frac{4}{3} \left(\frac{e_i(\boldsymbol{p}^{(t+1)},1) - e_i(\boldsymbol{p}^{(t)},1)}{e_i(\boldsymbol{p}^{(t)},1)} \right)^2 - \frac{e_i(\boldsymbol{p}^{(t+1)},1) - e_i(\boldsymbol{p}^{(t)},1)}{e_i(\boldsymbol{p}^{(t)},1)}$. By Lemma 7.2.10, we know that $0 + (-1/4) \leq a \leq \frac{1}{12} + 1/4 \Leftrightarrow -1/4 \leq a \leq \frac{1}{3}$. We now use the bound $x \geq \log(1+x)$ for $x > -1$, with $x = a$ to get:

$$
\begin{aligned}
& b_i \log \left(1 + \frac{4}{3} \left(\frac{e_i(\boldsymbol{p}^{(t+1)},1) - e_i(\boldsymbol{p}^{(t)},1)}{e_i(\boldsymbol{p}^{(t)},1)} \right)^2 - \frac{e_i(\boldsymbol{p}^{(t+1)},1) - e_i(\boldsymbol{p}^{(t)},1)}{e_i(\boldsymbol{p}^{(t)},1)} \right) \\
& \leq b_i \left(\frac{4}{3} \left(\frac{e_i(\boldsymbol{p}^{(t+1)},1) - e_i(\boldsymbol{p}^{(t)},1)}{e_i(\boldsymbol{p}^{(t)},1)} \right)^2 - \frac{e_i(\boldsymbol{p}^{(t+1)},1) - e_i(\boldsymbol{p}^{(t)},1)}{e_i(\boldsymbol{p}^{(t)},1)} \right)
\end{aligned}
$$

Using a first order Taylor expansion of $e_i(\boldsymbol{p}^{(t)} + \Delta \boldsymbol{p}, 1)$ around $\boldsymbol{p}^{(t)}$, by Taylor's theorem (Graves, 1927), we have: $e_i(\boldsymbol{p}^{(t)} + \Delta \boldsymbol{p}, 1) = e_i(\boldsymbol{p}^{(t)}, 1) + \left\langle \nabla_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)}, 1), \Delta \boldsymbol{p} \right\rangle + 1/2 \left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle$ for some $c \in (0,1)$. Re-organizing terms around, we get $e_i(\boldsymbol{p}^{(t+1)}, 1) - e_i(\boldsymbol{p}^{(t)}, 1) = \left\langle \nabla_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)}, 1), \Delta \boldsymbol{p} \right\rangle + 1/2 \left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle$, which gives us:

$$
\begin{aligned}
&= b_i \left(\frac{4}{3} \left(\frac{\left\langle \nabla_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)}, 1), \Delta \boldsymbol{p} \right\rangle + 1/2 \left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)} \right)^2 - \right. \\
& \left. \frac{\left\langle \nabla_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)}, 1), \Delta \boldsymbol{p} \right\rangle + 1/2 \left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)} \right)
\end{aligned} \tag{7.33}
$$

Continuing, by Shepherd's lemma (Shephard, 2015), we have:

$$= \frac{4}{3} b_i \left(\frac{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \Delta \boldsymbol{p} \right\rangle + 1/2 \left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)} \right)^2 -$$

$$b_i \frac{\left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \Delta \boldsymbol{p} \right\rangle + 1/2 \left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)} \tag{7.34}$$

$$= \frac{4}{3} \frac{1}{b_i} \left(\frac{b_i \left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \Delta \boldsymbol{p} \right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)} + \frac{b_i/2 \left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)} \right)^2 -$$

$$\frac{b_i \left\langle \boldsymbol{h}_i(\boldsymbol{p}^{(t)}, 1), \Delta \boldsymbol{p} \right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)} - \frac{b_i/2 \left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)} \tag{7.35}$$

$$= \frac{4}{3} \frac{1}{b_i} \left(\left\langle \boldsymbol{d}_i(\boldsymbol{p}^{(t)}, 1), \Delta \boldsymbol{p} \right\rangle + \frac{b_i/2 \left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)} \right)^2 -$$

$$\left\langle \boldsymbol{d}_i(\boldsymbol{p}^{(t)}, 1), \Delta \boldsymbol{p} \right\rangle - \frac{b_i/2 \left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)} \tag{7.36}$$

where the last line was obtained from Lemma 7.2.9.

Using Lemma 7.2.11, we have:

$$\leq \frac{4}{3} \left(\left(1 + \frac{5\epsilon}{9} \right) \sum_{l \in [m]} \frac{d_{il}^{(t)}}{p_l^{(t)}} (\Delta p_l)^2 + \frac{25\epsilon^2}{432} \sum_{l \in [m]} \frac{(\Delta p_l)^2}{p_l^{(t)}} d_{il}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i) \right) -$$

$$\left\langle \boldsymbol{d}_i(\boldsymbol{p}^{(t)}, 1), \Delta \boldsymbol{p} \right\rangle - \frac{b_i/2 \left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)} \tag{7.37}$$

$$= \left(\frac{4}{3} + \frac{20\epsilon}{27} \right) \sum_{l \in [m]} \frac{d_{il}^{(t)}}{p_l^{(t)}} (\Delta p_l)^2 + \frac{25\epsilon^2}{324} \sum_{l \in [m]} \frac{(\Delta p_l)^2}{p_l^{(t)}} d_{il}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i) -$$

$$\left\langle \boldsymbol{d}_i(\boldsymbol{p}^{(t)}, 1), \Delta \boldsymbol{p} \right\rangle - \frac{b_i/2 \left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)} \tag{7.38}$$

Finally, we note that $\nabla^2_{\boldsymbol{p}} e_i$ is negative semi-definite, meaning that we have $\left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle \leq 0$, allowing us to re-express Equation (7.38) as follows:

$$= \left(\frac{4}{3} + \frac{20\epsilon}{27} \right) \sum_{l \in [m]} \frac{d_{il}^{(t)}}{p_l^{(t)}} (\Delta p_l)^2 + \frac{25\epsilon^2}{324} \sum_{l \in [m]} \frac{(\Delta p_l)^2}{p_l^{(t)}} d_{il}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i) -$$

$$\left\langle \boldsymbol{d}_i(\boldsymbol{p}^{(t)}, 1), \Delta \boldsymbol{p} \right\rangle + \left| \frac{b_i/2 \left\langle \nabla^2_{\boldsymbol{p}} e_i(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, 1) \Delta \boldsymbol{p}, \Delta \boldsymbol{p} \right\rangle}{e_i(\boldsymbol{p}^{(t)}, 1)} \right| \tag{7.39}$$

$$\leq \left(\frac{4}{3} + \frac{20\epsilon}{27}\right) \sum_{l \in [m]} \frac{d_{il}^{(t)}}{p_l^{(t)}} (\Delta p_l)^2 + \frac{25\epsilon^2}{324} \sum_{l \in [m]} \frac{(\Delta p_l)^2}{p_l^{(t)}} d_{il}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i) - \left\langle \boldsymbol{d}_i(\boldsymbol{p}^{(t)}, 1), \Delta \boldsymbol{p} \right\rangle +$$

$$\frac{5\epsilon}{6} \sum_{l \in [m]} \frac{(\Delta p_l)^2}{p_l} d_{il}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i) \tag{7.40}$$

$$\leq \left(\frac{4}{3} + \frac{20\epsilon}{27}\right) \sum_{l \in [m]} \frac{d_{il}^{(t)}}{p_l^{(t)}} (\Delta p_l)^2 +$$

$$\left(\frac{5\epsilon}{6} + \frac{25\epsilon^2}{324}\right) \sum_{l \in [m]} \frac{(\Delta p_l)^2}{p_l^{(t)}} d_{il}(\boldsymbol{p}^{(t)} + c\Delta \boldsymbol{p}, b_i) - \left\langle \boldsymbol{d}_i(\boldsymbol{p}^{(t)}, 1), \Delta \boldsymbol{p} \right\rangle \tag{7.41}$$

where the penultimate line was obtained from Lemma 6.4.2.

Lemma 6.5.2 [Bounded Indirect Utility for Homothetic Fisher Markets].

If entropic *tâtonnement* is run on a homothetic Fisher market $(\boldsymbol{u}, \boldsymbol{b})$, then, for all $t \in \mathbb{N}_+$, the following bound holds:

$$v_i(\boldsymbol{p}^{(t)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) \leq v_i(\boldsymbol{p}^{(0)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) + 2 \max_{\substack{\boldsymbol{p}, \boldsymbol{q} \in \Delta_m \\ k \in [m]: h_{ik}(\boldsymbol{p}, 1) > 0}} \left\{ \frac{h_{ij}(\boldsymbol{q}, 1)^2}{h_{ik}(\boldsymbol{p}, 1)^2} \right\}$$

Proof of Lemma 6.5.2

Fix a buyer $i \in [n]$. First, note that since by Lemma 7.2.1, since the expenditure function is homogeneous of degree 0 in prices, we have for all $j \in [m]$, $\max_{\boldsymbol{p} \in \mathbb{R}_+^m} h_{ij}(\boldsymbol{p}, 1) = \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1)$. In addition, note that $\max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1)$ is well-defined since Δ_m is compact, $h_{ij}(\boldsymbol{p}, 1)$ exists for all $\boldsymbol{p} \in \mathbb{R}_+^m$, and is by Berge's maximum theorem (Berge, 1997) continuous in homothetic Fisher markets. We will now prove that for any $t \in \mathbb{N}$, $v_i(\boldsymbol{p}^{(t)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) \leq v_i(\boldsymbol{p}^{(0)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) + 2 \max_{\substack{\boldsymbol{p}, \boldsymbol{q} \in \Delta_m \\ k \in [m]: h_{ik}(\boldsymbol{p}, 1) > 0}} \left\{ \frac{h_{ij}(\boldsymbol{q}, 1)^2}{h_{ik}(\boldsymbol{p}, 1)^2} \right\}$ by induction.

Base case: $t = 0$**.** By definition, we have

$$v_i(\boldsymbol{p}^{(0)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) \leq v_i(\boldsymbol{p}^{(0)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) + 2 \max_{\substack{\boldsymbol{p}, \boldsymbol{q} \in \Delta_m \\ k \in [m]: h_{ik}(\boldsymbol{p}, 1) > 0}} \left\{ \frac{h_{ij}(\boldsymbol{q}, 1)^2}{h_{ik}(\boldsymbol{p}, 1)^2} \right\} .$$

Inductive hypothesis. Suppose that for any $t \in \mathbb{N}$, we have:

$$v_i(\boldsymbol{p}^{(t)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) \leq v_i(\boldsymbol{p}^{(0)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) + 2 \max_{\substack{\boldsymbol{p}, \boldsymbol{q} \in \Delta_m \\ k \in [m]: h_{ik}(\boldsymbol{p}, 1) > 0}} \left\{ \frac{h_{ij}(\boldsymbol{q}, 1)^2}{h_{ik}(\boldsymbol{p}, 1)^2} \right\}$$

Inductive step. We will show that the inductive hypothesis holds for $t+1$. We proceed with a proof by cases.

Case 1: $d_{ij}^{(t)} \geq \max_{\substack{\boldsymbol{p},\boldsymbol{q} \in \Delta_m \\ k \in [m]: h_{ik}(\boldsymbol{p},1) > 0}} \left\{ \frac{h_{ij}(\boldsymbol{q},1)}{h_{ik}(\boldsymbol{p},1)} \right\}$.

For all $k \in [m]$, we have:

$$\begin{aligned} d_{ik}^{(t)} &= \frac{h_{ik}^{(t)}}{h_{ij}^{(t)}} d_{ij}^{(t)} \\ &\geq \frac{h_{ik}^{(t)}}{h_{ij}^{(t)}} \max_{\substack{\boldsymbol{p},\boldsymbol{q} \in \Delta_m \\ k \in [m]: h_{ik}(\boldsymbol{p},1) > 0}} \left\{ \frac{h_{ij}(\boldsymbol{q},1)}{h_{ik}(\boldsymbol{p},1)} \right\} \\ &\geq \frac{h_{ik}^{(t)}}{h_{ij}^{(t)}} \frac{h_{ij}^{(t)}}{h_{ik}^{(t)}} = 1 \end{aligned}$$

where the penultimate line follows from the case hypothesis.

The above means that the price of all goods will increase in the next time period, i.e., $\forall k \in [m], p_k^{(t+1)} \geq p_k^{(t)}$ which implies that $e_i(\boldsymbol{p}^{(t+1)}, 1) \geq e_i(\boldsymbol{p}^{(t)}, 1) \geq 0$. In addition, note that the expenditure is positive since prices reach 0 only asymptotically under entropic *tâtonnement*. Which gives us:

$$\begin{aligned} \frac{b_i}{e_i(\boldsymbol{p}^{(t+1)}, 1)} &\leq \frac{b_i}{e_i(\boldsymbol{p}^{(t)}, 1)} \\ v_i(\boldsymbol{p}^{(t+1)}, b_i) &\leq v_i(\boldsymbol{p}^{(t)}, b_i) && \text{(Corollary 6.2.1)} \end{aligned}$$

Multiplying both sides by $\max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1)$, we have for all $j \in [m]$:

$$\begin{aligned} v_i(\boldsymbol{p}^{(t+1)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) &\leq v_i(\boldsymbol{p}^{(t)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) \\ &= v_i(\boldsymbol{p}^{(0)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) + 2 \max_{\substack{\boldsymbol{p},\boldsymbol{q} \in \Delta_m \\ k \in [m]: h_{ik}(\boldsymbol{p},1) > 0}} \left\{ \frac{h_{ij}(\boldsymbol{q},1)^2}{h_{ik}(\boldsymbol{p},1)^2} \right\} \end{aligned}$$

where the last line follows by the induction hypothesis.

Case 2: $d_{ij}^{(t)} < \max_{\substack{\boldsymbol{p},\boldsymbol{q} \in \Delta_m \\ k \in [m]: h_{ik}(\boldsymbol{p},1) > 0}} \left\{ \frac{h_{ij}(\boldsymbol{q},1)}{h_{ik}(\boldsymbol{p},1)} \right\}$.

For all $k \in [m]$, we have:

$$\begin{aligned} d_{ik}^{(t)} &= \frac{h_{ik}^{(t)}}{h_{ij}^{(t)}} d_{ij}^{(t)} \\ &\leq \frac{h_{ik}^{(t)}}{h_{ij}^{(t)}} \max_{\substack{\boldsymbol{p}, \boldsymbol{q} \in \Delta_m \\ k \in [m] : h_{ik}(\boldsymbol{p}, 1) > 0}} \left\{ \frac{h_{ij}(\boldsymbol{q}, 1)}{h_{ik}(\boldsymbol{p}, 1)} \right\} \\ &= \frac{h_{ik}^{(t)}}{h_{ij}^{(t)}} \frac{h_{ij}^{(t)}}{h_{ik}^{(t)}} = 1 \end{aligned}$$

where the penultimate line follows from the case hypothesis.

The above means that prices of all goods will decrease in the next time period. Now, note that regardless of the aggregate demand $\boldsymbol{q}^{(t)}$ at time $t \in \mathbb{N}$, prices can decrease at most by a factor of $e^{-\frac{1}{5}} \geq 1/2$, that is, for all $j \in [m]$

$$\begin{aligned} p_j^{(t+1)} &= p_j^{(t)} \exp \left\{ \frac{z_j(\boldsymbol{p}^{(t)})}{\gamma} \right\} \\ &= p_j^{(t)} \exp \left\{ \frac{q_j^{(t)} - 1}{\gamma} \right\} \\ &\geq p_j^{(t)} \exp \left\{ \frac{-1}{\gamma} \right\} \\ &\geq p_j^{(t)} \exp \left\{ \frac{-1}{5 \max_{\substack{t \in \mathbb{N} \\ j \in [m]}} \{1, q_j^{(t)}\}} \right\} \\ &\geq p_j^{(t)} \exp \left\{ \frac{-1}{5} \right\} \\ &\geq p_j^{(t)} e^{-\frac{1}{5}} \geq \frac{1}{2} p_j^{(t)} \end{aligned}$$

Now, notice that we have $e_i(\boldsymbol{p}^{(t+1)}, 1) \geq e_i(\frac{1}{2}\boldsymbol{p}^{(t)}, 1) = \frac{1}{2} e_i(\boldsymbol{p}^{(t)}, 1) \geq 0$, since the expenditure of the buyer increases the most when the prices of all goods decrease simultaneously and the expenditure function is homogeneous of degree 1 in prices. In addition, note that the expenditure is positive since prices reach 0 only asymptotically under entropic *tâtonnement*. Hence, we have:

$$\frac{b_i}{e_i(\boldsymbol{p}^{(t+1)}, 1)} \leq 2\frac{b_i}{e_i(\boldsymbol{p}^{(t)}, 1)}$$

$$v_i(\boldsymbol{p}^{(t+1)}, b_i) \leq 2v_i(\boldsymbol{p}^{(t)}, b_i) \qquad \text{(Corollary 6.2.1)}$$

Multiplying both sides by $h_{ij}^{(t)}$, and applying Lemma 7.2.9, we have for all $j \in [m]$:

$$v_i(\boldsymbol{p}^{(t+1)}, b_i)h_{ij}^{(t)} \leq 2d_{ij}^{(t)}$$

$$v_i(\boldsymbol{p}^{(t+1)})h_{ij}^{(t)} \leq 2 \max_{\substack{\boldsymbol{p},\boldsymbol{q} \in \Delta_m \\ k \in [m]: h_{ik}(\boldsymbol{p},1)>0}} \left\{ \frac{h_{ij}(\boldsymbol{q}, 1)}{h_{ik}(\boldsymbol{p}, 1)} \right\} \qquad \text{(Case hypothesis)}$$

Now, taking a maximum over all $j \in [m]$ s.t. $h_{ij}^{(t)} > 0$, we have

$$v_i(\boldsymbol{p}^{(t+1)}) \max_{k \in [m]: h_{ik}^{(t)}>0} h_{ik}^{(t)} \leq 2 \max_{\substack{\boldsymbol{p},\boldsymbol{q} \in \Delta_m \\ k \in [m]: h_{ik}(\boldsymbol{p},1)>0}} \left\{ \frac{h_{ij}(\boldsymbol{q}, 1)}{h_{ik}(\boldsymbol{p}, 1)} \right\}$$

$$v_i(\boldsymbol{p}^{(t+1)}) \max_{\substack{\boldsymbol{p} \in \Delta_m \\ k \in [m]: h_{ik}(\boldsymbol{p},1)>0}} h_{ik}(\boldsymbol{p}, 1) \leq 2 \max_{\substack{\boldsymbol{p},\boldsymbol{q} \in \Delta_m \\ k \in [m]: h_{ik}(\boldsymbol{p},1)>0}} \left\{ \frac{h_{ij}(\boldsymbol{q}, 1)}{h_{ik}(\boldsymbol{p}, 1)} \right\}$$

$$v_i(\boldsymbol{p}^{(t+1)}) \leq \frac{2}{\max_{\substack{\boldsymbol{p} \in \Delta_m \\ k \in [m]: h_{ik}(\boldsymbol{p},1)>0}} h_{ik}(\boldsymbol{p}, 1)} \max_{\boldsymbol{p},\boldsymbol{q}} \max_{k \in [m]: h_{ik}(\boldsymbol{p},1)>0} \left\{ \frac{h_{ij}(\boldsymbol{q}, 1)}{h_{ik}(\boldsymbol{p}, 1)} \right\}$$

Finally, multiplying both sides by $\max_{\boldsymbol{q} \in \Delta_m} h_{ij}(\boldsymbol{q}, 1)$, we have:

$$v_i(\boldsymbol{p}^{(t+1)}) \max_{\boldsymbol{q} \in \Delta_m} h_{ij}(\boldsymbol{q}, 1) \leq 2\frac{\max_{\boldsymbol{q} \in \Delta_m} h_{ij}(\boldsymbol{q}, 1)}{\max_{\substack{\boldsymbol{p} \in \Delta_m \\ k \in [m]: h_{ik}(\boldsymbol{p},1)>0}} h_{ik}(\boldsymbol{p}, 1)} \max_{\boldsymbol{p},\boldsymbol{q}} \max_{k \in [m]: h_{ik}(\boldsymbol{p},1)>0} \left\{ \frac{h_{ij}(\boldsymbol{q}, 1)}{h_{ik}(\boldsymbol{p}, 1)} \right\}$$

$$v_i(\boldsymbol{p}^{(t+1)}) \max_{\boldsymbol{q} \in \Delta_m} h_{ij}(\boldsymbol{q}, 1) \leq 2 \max_{\substack{\boldsymbol{p},\boldsymbol{q} \in \Delta_m \\ k \in [m]: h_{ik}(\boldsymbol{p},1)>0}} \left\{ \frac{h_{ij}(\boldsymbol{q}, 1)}{h_{ik}(\boldsymbol{p}, 1)} \right\}^2$$

$$v_i(\boldsymbol{p}^{(t+1)}) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) \leq v_i(\boldsymbol{p}^{(0)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) + 2 \max_{\substack{\boldsymbol{p},\boldsymbol{q} \in \Delta_m \\ k \in [m]: h_{ik}(\boldsymbol{p},1)>0}} \left\{ \frac{h_{ij}(\boldsymbol{q}, 1)^2}{h_{ik}(\boldsymbol{p}, 1)^2} \right\}$$

Hence, the inductive hypothesis holds for $t + 1$. Putting it all together, we have, for all $t \in \mathbb{N}$:

$$\sum_{i \in [n]} v_i(\boldsymbol{p}^{(t)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) \leq \sum_{i \in [n]} v_i(\boldsymbol{p}^{(0)}, b_i) \max_{\boldsymbol{p} \in \Delta_m} h_{ij}(\boldsymbol{p}, 1) + 2 \sum_{i \in [n]} \max_{\substack{\boldsymbol{p},\boldsymbol{q} \in \Delta_m \\ k \in [m]: h_{ik}(\boldsymbol{p},1)>0}} \left\{ \frac{h_{ij}(\boldsymbol{q}, 1)^2}{h_{ik}(\boldsymbol{p}, 1)^2} \right\}$$

Part II

Pseudo-Games and Arrow-Debreu Exchange Economies

Chapter 8

Scope and Motivation

8.1 Scope

Part II of this thesis is divided into two chapters. In Chapter 9, after reviewing background material on pseudo-games, I will introduce three new algorithmic approaches for computing equilibria in pseudo-games with polynomial-time guarantees. The first approach consists of a family of first-order algorithms known as mirror extragradient learning dynamics. I will prove that these methods converge to a variational equilibrium (VE) in variationally stable concave pseudo-games with jointly convex constraints. Beyond concave settings, I will establish convergence to a first-order variational equilibrium. Next, I will introduce two types of merit function minimization methods—one first-order and one second-order—that compute a solution satisfying the necessary conditions for a variational equilibrium in Lipschitz-smooth pseudo-games with jointly convex constraints.

In Chapter 10, after reviewing the foundational model of Arrow-Debreu economies, I will demonstrate that the set of Arrow-Debreu equilibria in any pure exchange economy corresponds exactly to the set of generalized Nash equilibria (GNE) of an associated variationally stable pseudo-game with jointly convex constraints. Leveraging this equivalence, I will introduce a novel family of market dynamics, called mirror extratrade dynamics, and prove their polynomial-time convergence to an Arrow-Debreu equilibrium in pure exchange economies. Finally, for more general, possibly non-concave, Arrow-Debreu economies, I will develop two polynomial-time merit function methods that compute a solution satisfying the necessary conditions for an Arrow-Debreu equilibrium.

8.2 Motivation

Walrasian equilibrium (Arrow and Debreu, 1954; Walras, 1896), first studied by French economist Léon Walras in 1874, is the steady state of an economy—any system governed by supply and demand (Walras, 1896). Walras assumed that each producer in an economy would act so as to maximize its profit, while consumers would make decisions that maximize their preferences over their available consumption choices; all this, while perfect competition prevails, meaning producers and consumers are unable to influence the prices that emerge. Under these assumptions, the demand and supply of each commodity is a function of prices, as they are a consequence of the decisions made by the producers and consumers, having observed the prevailing prices. A Walrasian equilibrium then corresponds to prices that solve the system of simultaneous equations with demand on one side and supply on the other, i.e., prices at which supply meets demand. Unfortunately, Walras did not provide conditions that guarantee the existence of such a solution, and the question of whether such prices exist remained open until Arrow and Debreu's rigorous analysis of Walrasian equilibrium in their model of a competitive economy in the middle of last century (Arrow and Debreu, 1954).

The Arrow-Debreu model comprises a set of commodities; a set of firms, each deciding what quantity of each commodity to supply; and a set of consumers, each choosing a quantity of each commodity to demand in exchange for their endowment (Arrow and Debreu, 1954). Arrow and Debreu define an **Arrow-Debreu equilibrium** as a collection of consumptions, one per consumer, a collection of productions, one per firm, and prices, one per commodity, such that fixing equilibrium prices: (1) no consumer can increase their utility by deviating to an alternative affordable consumption, (2) no firm can increase profit by deviating to another production in their production set, and (3) the **aggregate demand** for each commodity (i.e., the sum of the commodity's consumption across all consumers) does not exceed to its **aggregate supply** (i.e., the sum of the commodity's production and endowment across firms and consumers, respectively), while the total value of the aggregate demand is equal to the total value of the aggregate supply, i.e., **Walras' law** holds. That is, an Arrow-Debreu equilibrium is a tuple which consists of a Walrasian equilibrium, and the associated utility-maximizing consumptions and profit-maximizing productions.

Arrow and Debreu proceeded to show that their competitive economy could be seen as an **abstract economy**, which today is better known as a **pseudo-game** (Arrow and Debreu, 1954; Facchinei and Kanzow, 2010a). A pseudo-game is a generalization of a game in which the actions taken by each player impact not only the other players' payoffs, as in games, but also their set of permissible actions. Pseudo-games generalize games, and hence are even more widely applicable. Some recently studied applications include adversarial classification (Bruckner et al., 2012; Bruckner and Scheffer, 2009), energy resource allocation (Hobbs and Pang, 2007; Jing-Yuan and Smeers, 1999), environmental protection (Breton et al., 2006; Krawczyk, 2005), cloud computing (Ardagna et al., 2017; 2011), ride sharing services ((Jeff) Ban et al., 2019), transportation (Stein and Sudermann-Merx, 2018), wireless and network communication (Han et al., 2011; Pang et al., 2008).

Arrow and Debreu proposed **generalized Nash equilibrium** as the solution concept for this model, an action profile from which no player can improve their payoff by unilaterally deviating to another action in the space of permissible actions determined by the actions of other players. Arrow and Debreu further showed that any competitive economy could be represented as a pseudo-game inhabited by a fictional auctioneer, who sets prices so as to buy and resell commodities at a profit, as well as consumers and producers, who respectively, choose utility-maximizing consumptions of commodities in the budget sets determined by the prices set by the auctioneer, and profit-maximizing productions at the prices set by the auctioneer. The elegance of the reduction from competitive economies to pseudo-games is rooted in a simple observation: the set of Arrow-Debreu equilibria of a competitive economy is equal to the set of generalized Nash equilibria of the associated pseudo-game, implying the existence of Arrow-Debreu (and hence Walrasian) equilibrium in competitive economies as a corollary of the existence of generalized Nash equilibria in pseudo-games, whose proof is a straightforward generalization of Nash's proof for the existence of Nash equilibria (Nash, 1950b).[1]

Following Arrow and Debreu's seminal existence result, the literature turned its attention to questions of 1. (economic) **efficiency**, i.e., under what assumptions are Arrow-Debreu equilibria Pareto-optimal? (Arrow, 1951a;b; Arrow and Nerlove, 1958; Arrow and Hurwicz, 1958; Balasko, 1975; Debreu, 1951a); 2. **uniqueness**,

[1] McKenzie (1959) would prove the existence of Walrasian equilibrium independently, but concurrently. Much of his work, however, has gone unrecognized perhaps because his proof technique does not depend on this fundamental relationship between competitive and abstract economies.

i.e., under what assumptions are Arrow-Debreu equilibria unique? (Dierker, 1982; Pearce and Wise, 1973); 3. **stability**, i.e., under what conditions can a competitive economy settle into an Arrow-Debreu equilibrium? (Hahn, 1958; Balasko, 1975; Arrow and Hurwicz, 1958; Cole and Fleischer, 2008; Cheung et al., 2018; 2013; Goktas et al., 2023b), and 4. **efficient computation**, i.e., under what conditions can an Arrow-Debreu equilibrium be computed efficiently? (Jain et al., 2005; Codenotti et al., 2005; 2006; Chen and Teng, 2009).

In this part of the thesis, we seek to provide an answer to the latter two questions by introducing a family of algorithms to compute a generalized Nash equilibrium in pseudo-games. Work in this direction is progressing; see, for example, (Facchinei et al., 2009; Facchinei and Kanzow, 2010a; Facchinei and Sagratella, 2011; Paccagnan et al., 2016; Yi and Pavel, 2017; Couzoudis and Renner, 2013; Dreves, 2017; Von Heusinger and Kanzow, 2009; Tatarenko and Kamgarpour, 2018; Dreves and Sudermann-Merx, 2016; Von Heusinger et al., 2012; Izmailov and Solodov, 2014; Fischer et al., 2016; Pang and Fukushima, 2005; Facchinei and Lampariello, 2011; Fukushima, 2011; Kanzow, 2016; Kanzow and Steck, 2016; 2018; Ba and Pang, 2020). Nonetheless, there are still few, if any (Jordan et al., 2022), GNE-finding algorithms with computational guarantees, even for restricted classes of pseudo-games. We then apply the algorithms we developed to the computation of an Arrow-Debreu equilibrium in competitive economies, providing broad conditions under which Arrow-Debreu equilibria can be computed efficiently.

8.3 Contributions

8.3.1 Pseudo-Games

In Chapter 4, I advance the study of pseudo-games by refining their solution concepts and analyzing their computational complexity. First, I re-establish the existence of variational equilibrium in quasiconcave pseudo-games with jointly convex constraints (Theorem 9.2.2). I then introduce the notion of first-order variational equilibrium, which I show exists in a broader class of pseudo-games—namely, smooth games with jointly convex constraints (Theorem 9.5.2). Next, I establish an equivalence between (first-order) variational equilibria of pseudo-games and strong solutions of variational inequalities (Lemma 9.4.1 and Lemma 9.6.1). This allows me to define a new class of pseudo-games—variationally stable pseudo-games with jointly convex constraints—for which a first-order variational equilibrium can be computed in polynomial time via a novel uncoupled learning dynamic called the mirror extragradient learning dynamics (Theorem 9.6.1). In the special case where the pseudo-game is also concave, this result extends to the computation of variational equilibrium in polynomial time via these learning dynamics (Theorem 9.4.1)—to the best of my knowledge, the broadest result of its kind in the literature. Finally, for more general pseudo-games with jointly convex constraints that are not necessarily variationally stable, I develop two polynomial-time globally convergent merit function methods that compute a solution satisfying the necessary conditions for a variational equilibrium (Theorem 9.4.3 and Theorem 9.6.2).

8.3.2 Arrow-Debreu Economies

In Chapter 10, I provide novel mathematical characterizations of Arrow-Debreu equilibrium in Arrow-Debreu economies. First, I re-establish that the set of Arrow-Debreu equilibria of any quasiconcave Arrow-Debreu economy coincides with the set of generalized Nash equilibria of the corresponding Arrow-Debreu pseudo-game (Lemma 10.2.1). However, as the Arrow-Debreu pseudo-game characterization is intractable, I introduce an alternative characterization: the set of Arrow-Debreu equilibria of any concave pure exchange economy corresponds to the set of generalized Nash equilibria of the trading post pseudo-game (Lemma 10.3.2), which is a variationally stable pseudoconcave pseudo-game with jointly convex constraints. I then apply the mirror extragradient learning dynamics to solve this pseudo-game, leading to a market dynamic I call mirror extratrade dynamics. While the trading post pseudo-game is not concave, I show that it is pseudoconcave, implying that an approximate first-order variational equilibrium can be computed in polynomial time. Moreover, asymptotically, the algorithm converges to a variational equilibrium of the trading post pseudo-game—and thus to an Arrow-Debreu equilibrium of the associated concave pure exchange economy (Theorem 10.3.1). Finally, for more general, possibly non-concave, Arrow-Debreu economies, I develop two polynomial-time globally convergent merit function methods that compute a solution satisfying the necessary conditions for an Arrow-Debreu equilibrium (Theorem 10.4.1 and Theorem 10.4.2).

Chapter 9

Pseudo-games

9.1 Background

A **pseudo-game** (Arrow and Debreu, 1954) $(n, l, \mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$, denoted $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ when clear from context, comprises $n \in \mathbb{N}_+$ players, each player $i \in [n]$ can take an **action** $\boldsymbol{a}_i \in \mathcal{A}_i$ from its **action space** $\mathcal{A}_i$. An ordered tuple of per-player actions $\boldsymbol{a} \doteq (\boldsymbol{a}_1, \ldots, \boldsymbol{a}_n) \in \mathcal{A}$ is called an **action profile** where we define $\mathcal{A} \doteq \bigtimes_{i\in[n]} \mathcal{A}_i \subset \mathbb{R}^{nm}$ to be the space of action profiles. We denote any action profile $\boldsymbol{a} \in \mathcal{A}$ where the ith player's action is removed by $\boldsymbol{a}_{-i} \in \mathcal{A}_{-i}$ where $\mathcal{A}_{-i} \doteq \bigtimes_{i'\neq i} \mathcal{A}_{i'} \subset \mathbb{R}^{(n-1)m}$. Additionally, we use the notation $(\boldsymbol{a}'_i, \boldsymbol{a}_{-i}) \in \mathcal{A}$ to denote the action profile $\boldsymbol{a} \in \mathcal{A}$ where the ith player's action was replaced by the action $\boldsymbol{a}_i \in \mathcal{A}_i$ of i.

Each player i simultaneously chooses a **feasible** action from the set $\mathcal{X}_i(\boldsymbol{a}_{-i}) = \{\boldsymbol{a}_i \in \mathcal{A}_i \mid h_{ic}(\boldsymbol{a}_i, \boldsymbol{a}_{-i}) \geq 0, \text{ for all } c \in [l]\}$, determined by the actions $\boldsymbol{a}_{-i} \in \mathcal{A}_{-i}$ of the other players, where $h_{ic} : \mathcal{A} \to \mathbb{R}^l$ is the **action constraint function**, l is the number of constraints, and $\mathcal{X}_i : \mathcal{A}_{-i} \rightrightarrows \mathcal{A}_i$ is the **feasible action correspondence**. We denote the product of these feasible action correspondences by $\mathcal{X}(\boldsymbol{a}) = \bigtimes_{i\in[n]} \mathcal{X}_i(\boldsymbol{a}_{-i})$, and the **space of feasible action profiles** $\mathcal{X}^* \doteq \{\boldsymbol{a} \in \mathcal{A} \mid \boldsymbol{a} \in \mathcal{X}(\boldsymbol{a})\}$. Once players have taken a feasible action $\boldsymbol{a} \in \mathcal{X}(\boldsymbol{a})$, each player i receives a payoff $u_i(\boldsymbol{a})$ according to their **payoff function** $u_i : \mathcal{A} \to \mathbb{R}$. We denote **payoff profile function** by $\boldsymbol{u}(\boldsymbol{a}) \doteq (u_i(\boldsymbol{a}))_{i\in[n]}$.

9.2 Global Solution Concepts and Existence

9.2.1 Generalized Nash Equilibrium and Variational Equilibrium

The canonical solution concept for a pseudo-game is the generalized Nash equilibrium (GNE).

Definition 9.2.1 [Generalized Nash equilibrium]**.**

Given $\varepsilon \geq 0$, a ε-**generalized Nash equilibrium (GNE)** is an action profile $\boldsymbol{a}^* \in \mathcal{X}(\boldsymbol{a}^*)$ s.t. for all $i \in [n]$ and $\boldsymbol{a}_i \in \mathcal{X}_i(\boldsymbol{a}^*_{-i})$:

$$u_i(\boldsymbol{a}^*) \geq u_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i}) - \varepsilon \ .$$

A 0-GNE is simply called a generalized Nash equilibrium (GNE).

The GNE computation problem can succinctly be written as solving the following n simultaneous quasi-optimization problems:

$$\forall i \in [n], \qquad \max_{\boldsymbol{a}_i \in \mathcal{X}_i(\boldsymbol{a}_{-i})} u_i(\boldsymbol{a}_i, \boldsymbol{a}_{-i})$$

An important refinement of GNE is the variational equilibrium (VE):

Definition 9.2.2 [Variational equilibrium]**.**

Given $\varepsilon \geq 0$, a ε-**variational equilibrium (VE)** is an action profile $\boldsymbol{a}^* \in \mathcal{X}(\boldsymbol{a}^*)$ s.t. for all action profiles $\boldsymbol{a} \in \mathcal{X}^*$:

$$\sum_{i \in [n]} \left[u_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i}) - u_i(\boldsymbol{a}^*) \right] \leq \varepsilon \ .$$

A 0-VE is simply called a variational equilibrium (VE).

A really important class of pseudo-games are those which are unconstrained, more commonly known as games.

Definition 9.2.3 [Games and Nash equilibrium]**.**

A **game** (Nash, 1950b) $(n, \boldsymbol{u}, \mathcal{A})$, denoted $(\boldsymbol{u}, \mathcal{A})$ when clear from context, is a pseudo-game $(n, l, \mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ where $l \doteq 0$ i.e., the pseudo-game is unconstrained and there is no constraint function.

A ε-generalized Nash equilibrium of a game is simply called a ε-**Nash equilibrium** (**NE**). A 0-Nash equilibrium is simply called a Nash equilibrium.

Similar to the GNE computation problem, the NE computation problem for a game $(\mathcal{A}, \boldsymbol{u})$ can be succinctly expressed the following simultaneous optimization problem:

$$\forall i \in [n], \qquad \max_{\boldsymbol{a}_i \in \mathcal{A}_i} u_i(\boldsymbol{a}_i, \boldsymbol{a}_{-i})$$

Remark 9.2.1 [Relationship between GNE, VE and NE]**.**
While in general, the set of VE is a subset of the set of GNE in pseudo-games; in games the set of GNE is equal to the set of VE since $\mathcal{X}^* = \mathcal{A}$. The algorithms we will provide in this thesis will mostly be relevant to VEs, however due to the aforementioned relationship between solution concepts, it will also directly provide computational results for GNE and NE.

9.2.2 Quasiconcave Pseudo-Games

With a definition of our solution concepts in hand, we now describe the classes of games in which they are guaranteed to exist. The canonical class of pseudo-games in which a GNE is guaranteed to exist in the class of quasiconcave games. Related and of greater interest in the computational literature is the class of concave pseudo-games which is a strict subset of the class of quasiconcave pseudo-games.

Definition 9.2.4 [Quasiconcave pseudo-games]**.**
A **quasiconcave** (resp. **concave**) **pseudo-game** is a pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ where for all players $i \in [n]$:

[Continuous payoffs]	u_i is continuous;
[(Quasi)concave payoffs]	$\boldsymbol{a}_i \mapsto u_i(\boldsymbol{a}_i, \boldsymbol{a}_{-i})$ is quasiconcave (resp. concave) for all $\boldsymbol{a}_{-i} \in \mathcal{A}_{-i}$;
[Convex constraints]	$\mathcal{X}_{-i}$ is continuous, non-empty-, compact-, and convex-valued;
[Convex action space]	$\mathcal{A}_i$ is non-empty, compact, and convex.

Remark 9.2.2 [Convex constraints]**.**
The third condition in the above definition (i.e., the convex constraints condition) can be translated in terms of the action constraint function $\boldsymbol{h}$ assuming that $\boldsymbol{h}$ is continuous and satisfies Slater's condition (as defined in Definition 9.2.6), see, the end of Section 2.9 for additional details.

The importance of concave pseudo-games is due to a seminal result of Arrow and Debreu (1954) which established that a GNE is guaranteed to exist in all quasiconcave pseudo-games. This proof of existence relies on a fixed-point argument which we provide a shorter version of here for completeness.

Theorem 9.2.1 [Lemma on abstract economies (Lemma 2.5 of Arrow and Debreu (1954)]**.**
A GNE is guaranteed to exist in all quasiconcave pseudo-games.

Proof of Theorem 9.2.1

Define the **best-response correspondence** of the game as: $\mathcal{BR}(\boldsymbol{a}) \doteq \bigtimes_{i \in [n]} \arg\max_{\boldsymbol{a}'_i \in \mathcal{X}_i(\boldsymbol{a}_{-i})} u_i(\boldsymbol{a}'_i, \boldsymbol{a}_{-i})$. Now, consider any fixed point $\boldsymbol{a}^* \in \mathcal{BR}(\boldsymbol{a}^*)$ of the best-response correspondence. Then, for all players $i \in [n]$, we have:

$$\boldsymbol{a}^*_i \in \arg\max_{\boldsymbol{a}'_i \in \mathcal{X}_i(\boldsymbol{a}^*_{-i})} u_i(\boldsymbol{a}'_i, \boldsymbol{a}^*_{-i})$$

That is, $\boldsymbol{a}^*$ is a GNE. Now, notice that by the maximum theorem (Berge, 1997), in quasiconcave games, the best-response correspondence is guaranteed to be upper-hemicontinuous, non-empty-, compact-, and convex-valued. Hence, the best-response correspondence satisfies the assumptions of the Kakutani-Glicksberg fixed point theorem (Theorem 2.4.1), and a fixed point, and thus a GNE, exists.

Another important class of pseudo-games is the class of pseudo-games with jointly convex constraints.

Definition 9.2.5 [Jointly convex constraints]**.**
A pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ is said to have jointly convex constraints iff:

[Non-emptiness]	there exists $\boldsymbol{a} \in \mathcal{A}$ s.t. $\boldsymbol{h}(\boldsymbol{a}) \geq \boldsymbol{0}$ (i.e., $\mathcal{X}^*$ is non-empty);
[Compactness]	$\boldsymbol{a} \mapsto \boldsymbol{h}(\boldsymbol{a})$ is continuous (i.e., $\mathcal{X}^*$ is compact);
[Convexity]	for all players $i \in [n]$, and $c \in [l]$ $\boldsymbol{a} \mapsto h_{ic}(\boldsymbol{a})$ is quasiconcave (i.e., $\mathcal{X}^*$ is convex).

Joint convexity of constraints is important as a VE is guaranteed to exist in all quasiconcave pseudo-games with jointly convex constraints. The proof of existence of VE was first provided by Rosen (1965) and analogously to the proof of existence of GNE, applies a fixed-point argument to a suitable best-response correspondence. We provide here a brief proof of existence and refer the reader to Facchinei and Kanzow (2010a) for additional context.

Theorem 9.2.2 [Theorem 1 of Rosen (1965)]**.**

A VE is guaranteed to exist in all quasiconcave pseudo-games with jointly convex constraints.

Proof of Theorem 9.2.1

Define the **VE best-response correspondence** as: $\mathcal{VBR}(\boldsymbol{a}) \doteq \arg\max_{\boldsymbol{a}'_i \in \mathcal{X}^*} \sum_{i \in [n]} u_i(\boldsymbol{a}'_i, \boldsymbol{a}_{-i})$. Now, consider any fixed point $\boldsymbol{a}^* \in \mathcal{VBR}(\boldsymbol{a}^*)$ of the VE best-response correspondence. Then, for all action profiles $\boldsymbol{a} \in \mathcal{X}^*$, and players $i \in [n]$, we have:

$$u_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i}) - u_i(\boldsymbol{a}^*) \leq \varepsilon \ \ .$$

That is, $\boldsymbol{a}^*$ is a GNE. Now, notice that by the maximum theorem (Berge, 1997), in quasiconcave games with jointly convex constraints, the VE best-response correspondence is guaranteed to be upper-hemicontinuous, non-empty-, compact-, and convex-valued. Hence, the best-response correspondence satisfies the assumptions of the Kakutani-Glicksberg fixed point theorem (Theorem 2.4.1), and a fixed point, and thus a VE, exists.

9.2.3 Nash Equilibrium and Generalized Nash Equilibrium Equivalence

While pseudo-games are a more practical modeling framework than games for mathematical analysis, for a large number of pseudo-games, the two models are equivalent. As solution methods for constrained optimization problems most often, excluding the rare cases where projection onto a constraint set can be computed in closed form, require one to solve an unconstrained penalized optimization problem, this equivalence is of great importance to solve pseudo-games in practice. In particular, for a large class of pseudo-games, it is possible to achieve a reduction from any n-player pseudo-game to a $2n$-player game under the following standard constraint qualification.

Definition 9.2.6 [Slater's condition]**.**

A pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ is said to satisfy **Slater's condition** iff for all players $i \in [n]$, $c \in [l]$, and $\boldsymbol{a}_{-i} \in \mathcal{A}_{-i}$, there exist a **Slater vector** $\widetilde{\boldsymbol{a}}_i \in \text{relint}(\mathcal{A}_i)$:

(Concave constraint function) $\quad \boldsymbol{a}_i \mapsto h_{ic}(\boldsymbol{a}_i, \boldsymbol{a}_{-i})$ is concave

(Weak Slater) $\quad$ if $\boldsymbol{a}_i \mapsto h_{ic}(\boldsymbol{a}_i, \boldsymbol{a}_{-i})$ is affine, then $h_{ic}(\tilde{\boldsymbol{a}}_i, \boldsymbol{a}_{-i}) \geq 0$;

(Strong slater) $\quad$ otherwise, $h_{ic}(\tilde{\boldsymbol{a}}_i, \boldsymbol{a}_{-i}) > 0$

The following theorem shows that for any pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ that satisfies Slater's condition the GNE problem can be reduced to the following NE problem:

$$\forall i \in [n], \quad \max_{\boldsymbol{a}_i \in \mathcal{A}_i} u_i(\boldsymbol{a}_i, \boldsymbol{a}_{-i}) + \left\langle \boldsymbol{\lambda}_i, \boldsymbol{h}_i(\boldsymbol{a}_i, \boldsymbol{a}_{-i}) \right\rangle \qquad \forall i \in [n], \quad \min_{\boldsymbol{\lambda}_i \in \mathbb{R}^l_+} u_i(\boldsymbol{a}_i, \boldsymbol{a}_{-i}) + \left\langle \boldsymbol{\lambda}_i, \boldsymbol{h}_i(\boldsymbol{a}_i, \boldsymbol{a}_{-i}) \right\rangle$$

Note that in the above, the objective problems of the minimization problems should be interpreted as negated payoffs in the game.

Theorem 9.2.3 [Pseudo-game to game reduction]**.**

Consider a pseudo-game $(n, l, \mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ which satisfies Slater's condition. Define the game $(2n, \mathcal{A}', \boldsymbol{u}')$ where for $i \in [2n]$:

$$\text{(Action space)} \quad \mathcal{A}'_i \doteq \begin{cases} \mathcal{A}_i & \text{if } i \in [n] \\ \mathbb{R}^l_+ & \text{otherwise} \end{cases}$$

$$\text{(Payoffs)} \quad u'_i(\boldsymbol{a}, \boldsymbol{\lambda}) \doteq \begin{cases} u_i(\boldsymbol{a}) + \langle \boldsymbol{\lambda}_i, \boldsymbol{h}_i(\boldsymbol{a}) \rangle & \text{if } i \in [n] \\ -u_{i-n}(\boldsymbol{a}) - \left\langle \boldsymbol{\lambda}_i, \boldsymbol{h}_{i-n}(\boldsymbol{a}) \right\rangle & \text{otherwise} \end{cases}$$

Then, $(\boldsymbol{a}^*, \boldsymbol{\lambda}^*) \in \mathcal{A} \times \mathbb{R}^l_+$ is a NE of $(2n, \mathcal{A}', \boldsymbol{u}')$ iff $\boldsymbol{a}^*$ is a GNE of $(2n, \mathcal{A}', \boldsymbol{u}')$.

Proof

$(\implies)$: Let $(\boldsymbol{a}^*, \boldsymbol{\lambda}^*) \in \mathcal{A} \times \mathbb{R}^l_+$ be a NE of $(2n, \mathcal{A}', \boldsymbol{u}')$. Then, for all $i \in [n]$, $(\boldsymbol{a}_i^*, \boldsymbol{\lambda}_i^*)$ is saddle point (i.e., NE) of the following min-max optimization problem

$$\max_{\boldsymbol{a}_i \in \mathcal{A}_i} \min_{\boldsymbol{\lambda} \in \mathbb{R}^l_+} u_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i}) + \left\langle \boldsymbol{\lambda}_i, \boldsymbol{h}_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i}) \right\rangle$$

Hence, by the KKT theorem (Kuhn and Tucker, 1951), for all $i \in [n]$, $\boldsymbol{a}_i^*$ is a solution of the optimization problem:

$$\max_{\boldsymbol{a}_i \in \mathcal{X}_i(\boldsymbol{a}_{-i})} u_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i})$$

That is, $\boldsymbol{a}^*$ is a GNE of $(2n, \mathcal{A}', \boldsymbol{u}')$.

($\Longleftarrow$): Let $\boldsymbol{a}^* \in \mathcal{X}(\boldsymbol{a}^*)$ be a GNE of $(2n, \mathcal{A}', \boldsymbol{u}')$. That is, for all players $i \in [n]$, $\boldsymbol{a}^*$ is a solution of:

$$\max_{\boldsymbol{a}_i \in \mathcal{X}_i(\boldsymbol{a}_{-i})} u_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i})$$

Since Slater's condition is satisfied, by the KKT theorem (Kuhn and Tucker, 1951), for all $i \in [n]$, there exists $\boldsymbol{\lambda}^*_i \in \mathbb{R}^l_+$ s.t. $(\boldsymbol{a}^*_i, \boldsymbol{\lambda}^*_i)$ is a solution of the following Langrangian saddle point problem:

$$\max_{\boldsymbol{a}_i \in \mathcal{A}_i} \min_{\boldsymbol{\lambda} \in \mathbb{R}^l_+} u_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i}) + \left\langle \boldsymbol{\lambda}_i, \boldsymbol{h}_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i}) \right\rangle$$

That is, $(\boldsymbol{a}^*, \boldsymbol{\lambda}^*) \in \mathcal{A} \times \mathbb{R}^l_+$ is a NE of $(2n, \mathcal{A}', \boldsymbol{u}')$.

Remark 9.2.3 [Boundedness of KKT multipliers]**.**

We note that under Slater's condition any KKT multiplier $\boldsymbol{\lambda}^*$ associated with the NE $(\boldsymbol{a}^*, \boldsymbol{\lambda}^*) \in \mathcal{A} \times \mathbb{R}^{nl}_+$ of the game $(2n, \mathcal{A}', \boldsymbol{u}')$ is guaranteed to be bounded. In particular, by Lemma 3 of Nedic and Ozdaglar (2009) we can define a a non-empty, compact, convex set $\Lambda \subseteq \mathbb{R}^{nl}_+$, whose diameter depends on the value of the utility functions of the players evaluated at the Slater vector of the game and which can be shown to contain the optimal KKT multipliers associated with the NE $(\boldsymbol{a}^*, \boldsymbol{\lambda}^*) \in \mathcal{A} \times \mathbb{R}^{nl}_+$ of the game $(2n, \mathcal{A}', \boldsymbol{u}')$.

With this reduction in hand, we will for the rest of this chapter focus on the computation of VE. Nevertheless, readers interested in applying the algorithms provided in this thesis can use the equivalence provided by the above theorem to compute a GNE in pseudo-games where a VE does not exists.

9.3 Algorithms for Pseudo-Games

9.3.1 Computational model

With the question of existence answered, we now turn our attention to the computation of VE (and thus GNE and NE) in concave pseudo-game. While a GNE is guaranteed to exist in quasiconcave pseudo-games, we will restrict our attention to concave pseudo-games as the computation of a ε-NE even in single player quasiconcave games (i.e., quasiconcave optimization) is known to be NP-hard (Vavasis, 1995).

Algorithms for the computation of GNE, VE, and NE can be categorized into two main categories, decentralized algorithms, called uncoupled learning dynamics in which players employ a learning algorithm independently from the others, and centralized algorithms ones which aim to compute an equilibrium without any restrictions.

Definition 9.3.1 [kth-order learning dynamics]**.**

Given some $k \in \mathbb{N}_{++}$, a pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ for which the derivatives $\{\nabla^j \boldsymbol{u}\}_{j=1}^{k-1}$ are well defined, and an initial iterate $\boldsymbol{a}^{(0)} \in \mathcal{A}$, a k**th-order learning dynamic** $\boldsymbol{\pi}$ consists of an update function which generates the sequence of iterates $\{\boldsymbol{a}^{(t)}\}_t$ given for all $t = 0, 1, \ldots$ by:

$$\boldsymbol{a}^{(t+1)} \doteq \boldsymbol{\pi}\left(\bigcup_{i=0}^{t}(\boldsymbol{a}^{(i)}, \{\nabla^j \boldsymbol{u}(\boldsymbol{a}^{(i)})\}_{j=0}^{k-1})\right)$$

The most prominent class of kth-order learning dynamics in the literature on equilibrium computation, are special class of first-order learning dynamics for games called uncoupled learning dynamics (Hart and Mas-Colell, 2003) (for a recent survey, see for instance, Golowich et al. (2020a)).

Definition 9.3.2 [Uncoupled learning dynamics for games]**.**

Given a game $(\mathcal{A}, \boldsymbol{u})$, and an initial action profile $\boldsymbol{a} \in \mathcal{A}$, an **(first-order) uncoupled learning dynamic** (Hart and Mas-Colell, 2003) $\boldsymbol{\pi} = (\boldsymbol{\pi}_1, \ldots, \boldsymbol{\pi}_n)$ consists of an update function $\boldsymbol{\pi}_i : \bigcup_{\tau \geq 1}(\mathcal{A}_i \times \mathbb{R} \times \mathcal{A}_i^*) \to \mathcal{A}_i$ for each player $i \in [n]$, which generates the sequence of actions $\{\boldsymbol{a}^{(t)}\}_t$ given for all player $i \in [n]$ and $t = 0, 1, \ldots$ by:

$$\boldsymbol{a}_i^{(t+1)} \doteq \boldsymbol{\pi}_i\left(\bigcup_{k=0}^{t}(\boldsymbol{a}_i^{(t)}, u_i(\boldsymbol{a}^{(t)}), \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^{(t)}))\right)$$

Remark 9.3.1 [Uncoupled Learning Dynamics in Games and Pseudo-games]**.**

Uncoupled learning dynamics in games can be understood as players playing the game repeatedly, and updating their action at each round based on the observations in prior rounds without coordinating with other players, thus the "uncoupled learning dynamics" terminology. By the constrained nature of pseudo-games, it is in general inappropriate to consider uncoupled learning dynamics for pseudo-games, since this would imply that players can play actions which are not feasible during the repeated pseudo-game.

As a remedy, one can generalize the notion of uncoupled learning dynamics for pseudo-games by introducing an "arbiter" who collects all players action updates, and then projects them back onto the space of feasible action profiles $\mathcal{X}^*$. To avoid making notation heavier, we will not introduce such a definition but we note that the mirror extragradient learning dynamics we will introduce for the computation of VE can be seen as such a type of an uncoupled learning dynamics. Justifying further this generalized definition, we note that when the first-order learning dynamics we introduce for pseudo-games are instead applied to games, as we will show the arising dynamics correspond to uncoupled learning dynamics. As a result, we will call a learning dynamic for pseudo-games "uncoupled" if when applied to a game the learning dynamics are uncoupled.

The computational complexity results in this chapter will rely on the following computational model which has been broadly adopted by the literature (see, for instance, Golowich et al. (2020a)).

Definition 9.3.3 [Pseudo-Game Computational Model]**.**
Given a pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$, and a kth-order learning dynamic $\boldsymbol{\pi}$, the computational complexity of a kth-order learning dynamic is measured in term of the number of evaluations of the the functions $\boldsymbol{u}, \nabla \boldsymbol{u}, \dots, \nabla^k \boldsymbol{u}$.

Remark 9.3.2.
In line with the literature, the computational model we consider thus assumes that any other operation such as (Bregman) projection onto a set is a constant cost operation.

The computational results that exist in the literature, as well as the results we will present in this chapter hold in the following two classes of pseudo-games.

Definition 9.3.4 [Lipschitz-Smooth Pseudo-Games]**.**
Given a modulus of smoothness $\lambda \geq 0$, a pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ is said to be **λ-Lipschitz-smooth** iff for all players $i \in [n]$, $\nabla_{\boldsymbol{a}_i} u_i$ is λ-Lipschitz-continuous.

Definition 9.3.5 [Jointly Lipschitz-Smooth Pseudo-Games]**.**
Given a modulus of smoothness $\lambda \geq 0$, a **jointly λ-Lipschitz-smooth** pseudo-game is a pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ such that for all players $i \in [n]$, u_i is λ-Lipschitz-smooth.

9.3.2 Related Works

Following Arrow and Debreu's introduction of GNE, Rosen (1965) initiated the study of the mathematical and computational properties of GNE in pseudo-games with jointly convex constraints, proposing a projected gradient method to compute GNE. Thirty years later, Uryas'ev and Rubinstein (1994) developed the first relaxation methods for finding GNEs, which were improved upon in subsequent works (Krawczyk and Uryasev, 2000; Heusinger and Kanzow, 2009). Two other types of algorithms were also introduced to the literature: Newton-style methods (Facchinei et al., 2009; Dreves, 2017; Von Heusinger et al., 2012; Izmailov and Solodov, 2014; Fischer et al., 2016; Dreves et al., 2013) and interior-point potential methods (Dreves et al., 2013). Many of these approaches are based on minimizing the exploitability of the pseudo-game, but others use variational inequality (Facchinei et al., 2007; Nabetani et al., 2011) and Lemke methods (Schiro et al., 2013).

More recently, novel methods that transform GNEP to NEP were analyzed. These models take the form of either exact penalization methods, which lift the constraints into the objective function via a penalty term (Facchinei and Lampariello, 2011; Fukushima, 2011; Kanzow and Steck, 2018; Ba and Pang, 2020; Facchinei and Kanzow, 2010b), or augmented Lagrangian methods (Pang and Fukushima, 2005; Kanzow, 2016; Kanzow and Steck, 2018; Bueno et al., 2019), which do the same, augmented by dual Lagrangian variables. Using these methods, Jordan et al. (2022) provide the first convergence rates to a ε-GNE in monotone (resp. strongly monotone) pseudo-games with jointly affine constraints in $\tilde{O}(1/\varepsilon)$ ($\tilde{O}(1/\sqrt{\varepsilon})$) iterations. These algorithms, despite being highly efficient in theory, are numerically unstable in practice (Jordan et al., 2022). Nearly all of the aforementioned approaches concerned pseudo-games with jointly convex constraints.

Exploitability minimization has also been a valuable tool in multi-agent reinforcement learning; algorithms in this literature that aim to minimize exploitability are known as exploitability-descent algorithms. Lockhart et al. (2019) analyzed exploitability descent in two-player, zero-sum, extensive-form games with finite action spaces. Variants of exploitability-descent have also been combined with entropic regularization and homotopy methods to solve for NE in large games (Gemp et al., 2021).

9.4 Computation of GNE

9.4.1 Uncoupled Learning Dynamics for GNE

The first type of algorithms we will study for the computation of VE are uncoupled learning dynamics. For convenience, going forward, we will define the following important operator.

Definition 9.4.1 [Pseudo-Game operator]**.**
The **pseudo-game operator** associated with any pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ is defined as

$$\boldsymbol{v}(\boldsymbol{a}) \doteq -(\nabla_{\boldsymbol{a}_1} u_1(\boldsymbol{a}), \dots, \nabla_{\boldsymbol{a}_n} u_n(\boldsymbol{a})) \enspace .$$

We denote the ith component of $\boldsymbol{v}(\boldsymbol{a})$ as $\boldsymbol{v}_i(\boldsymbol{a})$ which we note is equal to the negated gradient $-\nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a})$ of the ith player's payoff w.r.t. its own action.

Remark 9.4.1 [Generalization for subdifferentiable pseudo-games]**.**
In the above definition, for clarity it is assumed that $\nabla_{\boldsymbol{a}_i} u_i$ is well-defined. However, the definition can easily be extended to pseudo-games for which the subdifferential $\mathcal{D}_{\boldsymbol{a}_i} u_i$ is guaranteed to be non-empty, by defining the pseudo-game operator $\boldsymbol{v}$ as a correspondence, now better denoted $\mathcal{V}(\boldsymbol{a}) \doteq -\bigtimes_{i \in [n]} \mathcal{D}_{\boldsymbol{a}_i} u_i(\boldsymbol{a})$. The following lemma then also directly generalizes to such pseudo-games by replacing the VI $(\mathcal{X}^*, \boldsymbol{v})$ by the VI $(\mathcal{X}^*, \mathcal{V})$.

To introduce the uncoupled learning dynamic we will study, let us first introduce the following lemma which states that the ε-VE of any concave game can be expressed as the set of ε-strong solutions of the VI $(\mathcal{X}^*, \boldsymbol{v})$.

Lemma 9.4.1 [SVI = VE in Concave Pseudo-Games]**.**
The set of ε-VE of any concave pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ is equal to the set of ε-strong solutions $\mathcal{SVI}_\varepsilon(\mathcal{X}^*, \boldsymbol{v})$ of the VI $(\mathcal{X}^*, \boldsymbol{v})$.

Proof of Lemma 9.4.1

($\implies$) : Let $\boldsymbol{a}^* \in \mathcal{X}^*$ be a VE of $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$, then for all players $i \in [n]$ and $\boldsymbol{a} \in \mathcal{X}^*$, we have:

$$\begin{aligned}
\varepsilon &\geq \sum_{i \in [n]} u_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i}) - u_i(\boldsymbol{a}^*) \\
&\geq \sum_{i \in [n]} \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}_i, \boldsymbol{a}^*_i), \boldsymbol{a}_i - \boldsymbol{a}^*_i \right\rangle \\
&\geq \sum_{i \in [n]} \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}_i - \boldsymbol{a}^*_i \right\rangle = \langle \boldsymbol{v}(\boldsymbol{a}^*), \boldsymbol{a}^* - \boldsymbol{a} \rangle
\end{aligned}$$

where the penultimate and last lines were obtained by the assumption that the game is concave, which respectively guarantees that the payoff of the players are concave in their action, and the gradient of the payoff of the players is monotone.

Hence, $\boldsymbol{a}^*$ is a ε-strong solution of the VI $(\mathcal{X}^*, \boldsymbol{v})$.

($\impliedby$) : Let $\boldsymbol{a}^* \in \mathcal{X}^*$ be a ε-strong solutions of the VI $(\mathcal{X}^*, \boldsymbol{v})$, then we have for all $\boldsymbol{a} \in \mathcal{X}^*$:

$$\begin{aligned}
\varepsilon &\geq \langle \boldsymbol{v}(\boldsymbol{a}^*), \boldsymbol{a}^* - \boldsymbol{a} \rangle \\
&= \sum_{i \in [n]} \left\langle -\nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}^*_i - \boldsymbol{a}_i \right\rangle \\
&= \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}_i - \boldsymbol{a}^*_i \right\rangle \\
&= u_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i}) - u_i(\boldsymbol{a}^*)
\end{aligned}$$

Hence, $\boldsymbol{a}^*$ is a ε-VE of $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$.

Since the set of VE is equal to the set of NE in games, and $\mathcal{X}^* = \mathcal{A}$, we also have the following corollary of Lemma 9.4.1.

Corollary 9.4.1.

The set of ε-NE of any concave game $(\mathcal{A}, \boldsymbol{u})$ is equal to the set of ε-strong solutions of the VI $(\mathcal{A}, \boldsymbol{v})$.

With this lemma in hand, we can now apply the mirror extragradient method to solve the VI $(\mathcal{X}^*, \boldsymbol{v})$ associated with any concave pseudo-game $(\mathcal{A}, \boldsymbol{u})$ giving us the mirror extragradient learning dynamics (Algorithm 7). Here, we remark that when the pseudo-game considered is in fact a game, then the mirror extragradient method can be seen as an uncoupled learning dynamic.

Algorithm 7 Mirror Extragradient Learning Dyanmics

Input: $\mathcal{A}, \boldsymbol{h}, \boldsymbol{u}, \tau, \eta, h, \boldsymbol{a}^{(0)}$

Output: $\{\boldsymbol{a}^{(t)}, \boldsymbol{a}^{(t+0.5)}\}_{t\in[\tau]}$

1: **for** $t = 1, \ldots, \tau$ **do**

2: $\quad \forall i \in [n], \quad \boldsymbol{a}_i^{(t+0.5)} \leftarrow \arg\max_{\boldsymbol{a}_i\in\mathcal{A}_i} \left\{\left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^{(t)}), \boldsymbol{a}_i - \boldsymbol{a}_i^{(t)}\right\rangle - \frac{1}{2\eta}\mathrm{div}_h(\boldsymbol{a}_i, \boldsymbol{a}_i^{(t)})\right\}$

3: $\quad \forall i \in [n], \boldsymbol{a}_i^{(t+1)} \leftarrow \arg\max_{\boldsymbol{a}_i\in\mathcal{A}_i} \left\{\left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^{(t+0.5)}), \boldsymbol{a}_i - \boldsymbol{a}_i^{(t)}\right\rangle - \frac{1}{2\eta}\mathrm{div}_h(\boldsymbol{a}_i, \boldsymbol{a}_i^{(t)})\right\}$

return $\{\boldsymbol{a}^{(t)}, \boldsymbol{a}^{(t+0.5)}\}_{t\in[\tau]}$

Remark 9.4.2 [Mirror Extragradient Learning Dynamics Are Uncoupled]**.**

It turns out that the mirror extragradient learning dynamics (Algorithm 7) when applied to a game $(\mathcal{A}, \boldsymbol{u})$ can be seen as an uncoupled learning dynamic.

To see this, suppose that the mirror extragradient algorithm applied to $(\mathcal{A}, \boldsymbol{v})$ generates the sequence of action profiles $\{\boldsymbol{a}'^{(t)}, \boldsymbol{a}'^{(t+0.5)}\}_{t\in[0,\tau]}$. Now, for notational convenience, map the indices of the sequence through the transformation $t \mapsto 2t$ to obtain the sequence $\{\boldsymbol{a}^{(t)}\}_{t\in[0,2\tau]}$, then the transformed sequence of actions can be interpreted as an uncoupled learning dynamic where the update function for each player $i \in [n]$ and any $t \in \mathbb{N}$ is given by:

$$\boldsymbol{\pi}_i\left(\bigcup_{k=0}^{t}(\boldsymbol{a}_i^{(t)}, u_i(\boldsymbol{a}^{(t)}), \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^{(t)}))\right) \doteq \begin{cases} \arg\max_{\boldsymbol{a}_i\in\mathcal{A}_i} \left\{\left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^{(t)}), \boldsymbol{a}_i - \boldsymbol{a}_i^{(t)}\right\rangle - \frac{1}{2\eta}\mathrm{div}_h(\boldsymbol{a}_i, \boldsymbol{a}_i^{(t)})\right\} & \text{if } t \text{ is even} \\ \arg\max_{\boldsymbol{a}_i\in\mathcal{A}_i} \left\{\left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^{(t)}), \boldsymbol{a}_i - \boldsymbol{a}_i^{(t-1)}\right\rangle - \frac{1}{2\eta}\mathrm{div}_h(\boldsymbol{a}_i, \boldsymbol{a}_i^{(t-1)})\right\} & \text{if } t \text{ is odd} \end{cases} .$$

We note that the $\arg\min$ is singleton-valued when the kernel function h is strictly convex, and hence we interpret the $\arg\min$ as the item in the singleton output. The arising update rule then dictates that on even time steps all players take a step of mirror ascent updating their action the current time-step, while on odd time-steps, players take a step of mirror ascent updating their action on the previous time-step.

With this in mind, we now introduce the class of variationally stable games for which we can prove the convergence of the mirror extragradient learning dynamics. While a definition of variational stability was first introduced by Zhou et al. (2017) for games, the definition we provide here is much weaker than that considered by Zhou et al. (2017).[1] While Zhou et al. (2017) proved the asymptotic convergence of mirror ascent dynamics with decreasing step size in games when the kernel function h satisfies a set of regularity conditions, it is not clear if there exists a kernel function which satisfies these conditions and whether if this convergence implies a polynomial-time computation of a ε-NE in such games (or more broadly of ε-VE in pseudo-games. In contrast, here, we provide a non-asymptotic convergence rate under very mild assumptions on the kernel function h, namely strong convexity, which implies the polynomial-time computation of ε-VE in variationally stable pseudo-games.

Definition 9.4.2 [Variationally Stable Pseudo-Games]**.**

A pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ is said to be **variationally stable** iff there exists $\boldsymbol{a}^* \in \mathcal{X}^*$ s.t. for all $\boldsymbol{b} \in \mathcal{X}^*$:

$$\sum_{i \in [n]} \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{b}), \boldsymbol{a}_i^* - \boldsymbol{b}_i \right\rangle \geq 0$$

In other words, the pseudo-game is variationally stable iff the set of weak solutions $\mathcal{MVI}(\mathcal{X}^*, \boldsymbol{v})$ of the VI $(\mathcal{X}^*, \boldsymbol{v})$ is non-empty.

Remark 9.4.3 [Interpretting variational stability in concave pseudo-games]**.**

In concave games pseudo-games, by concavity, we have for all players $i \in [n]$, and $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{A}$:

$$u_i(\boldsymbol{a}_i, \boldsymbol{b}_{-i}) - u_i(\boldsymbol{b}) \leq \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{b}), \boldsymbol{a}_i - \boldsymbol{b}_i \right\rangle$$

Hence, the variational stability condition is satisfied if there exists $\boldsymbol{a}^* \in \mathcal{X}^*$ s.t. for all $i \in [n]$, $\boldsymbol{b} \in \mathcal{X}^*$:

$$u_i(\boldsymbol{a}_i^*, \boldsymbol{b}_{-i}) - u_i(\boldsymbol{b}) \geq 0$$

In other words, variational stability in concave pseudo-games can be ensured if there exists for all players $i \in [n]$ an action $\boldsymbol{a}_i^* \in \mathcal{X}^*$ that weakly increases its payoff when the player i unilaterally deviates from the action profile $\boldsymbol{b}$. In this sense, variational stability can be interpretted as the existence of a "first-order dominant strategy equilibrium".

The class of variationally stable concave pseudo-games contains a number of well-studied pseudo-games games such as monotone pseudo-games with jointly convex constraints.[2]

[1]When generalized for pseudo-games, per Zhou et al. a pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ is said to be variationally stable if its set of VE is equal to the set of weak solutions $\mathcal{MVI}(\mathcal{X}^*, \boldsymbol{v})$ of the VI $(\mathcal{X}^*, \boldsymbol{v})$. In contrast, to Zhou et al.'s definition, we only require the set of weak solution of the VI $(\mathcal{X}^*, \boldsymbol{v})$ to be non-empty, hence generalizing Zhou et al.'s definition.

[2]Note that the pseudo-game being monotone implies that for all players $i \in [n]$, and $\boldsymbol{a}_{-i} \in \mathcal{A}_{-i}$, $\boldsymbol{a}_i \mapsto u_i(\boldsymbol{a}_i, \boldsymbol{a}_{-i})$ is concave, however it does not imply continuity of u_i, and the non-emptyness, compactness, convexity of $\mathcal{X}^*$, which is necessary for the variational stability condition to hold.

Definition 9.4.3 [Monotone pseudo-games]**.**

A pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ is said to be **monotone** iff the pseudo-game operator $\boldsymbol{v}$ is monotone, i.e., for all $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{A}$,

$$\sum_{i \in [n]} \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}) - \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{b}), \boldsymbol{a}_i - \boldsymbol{b}_i \right\rangle \leq 0$$

As well as the larger but less studied class of pseudomonotone and quasimonotone concave pseudo-games with jointly convex constraints (see, for instance, Section 2.3.2 of Huang and Zhang (2023)):

Definition 9.4.4 [Pseudomonotone games]**.**

A pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ is said to be **pseudomonotone** iff the pseudo-game operator $\boldsymbol{v}$ is pseudomonotone, i.e., for all $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{A}$,

$$\sum_{i \in [n]} \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}), \boldsymbol{b}_i - \boldsymbol{a}_i \right\rangle \leq 0 \implies \sum_{i \in [n]} \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{b}), \boldsymbol{b}_i - \boldsymbol{a}_i \right\rangle \leq 0$$

Definition 9.4.5 [Quasimonotone Pseudo-Games]**.**

A game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ is said to be **quasimonotone** iff the game operator is quasimonotone $\boldsymbol{v}$, i.e., for all $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{A}$,

$$\sum_{i \in [n]} \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}), \boldsymbol{b}_i - \boldsymbol{a}_i \right\rangle < 0 \implies \sum_{i \in [n]} \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{b}), \boldsymbol{b}_i - \boldsymbol{a}_i \right\rangle \leq 0$$

With these definitions in hand, we can obtain a non-asymptotic convergence rate for the mirror extragradient learning dynamics in variationally stable pseudo-games as an application of Theorem 4.3.1 in conjunction with Lemma 9.4.1.

Theorem 9.4.1 [Convergence of Mirror Extragradient Learning Dynamics]**.**

Let $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ be a variationally stable and λ-Lipschitz-smooth concave pseudo-game with jointly convex constraints, and h a 1-strongly-convex and κ-Lipschitz-smooth kernel function. Consider the mirror extragradient learning dynamics (Algorithm 7) run with the pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$, the kernel function h, a step size $\eta \in \left(0, \frac{1}{\sqrt{2}\lambda}\right]$, for any time horizon $\tau \in \mathbb{N}$, and outputs $\{\boldsymbol{x}^{(t+0.5)}, \boldsymbol{x}^{(t+1)}\}_t$. Let $\boldsymbol{a}^{(\tau)}_{\text{best}} \in \arg\min_{\boldsymbol{x}^{(k+0.5)}: k=0,\ldots,\tau} \text{div}_h(\boldsymbol{a}^{(k+0.5)}, \boldsymbol{a}^{(k)})$. Then, for some choice of $\tau \in O(\frac{1}{\varepsilon^2})$, $\boldsymbol{a}^{(\tau)}_{\text{best}}$ is a ε-VE of $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$. In addition, the iterates generated asymptotically converge to some VE $\boldsymbol{a}^* \in \mathcal{X}^*$ of $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$, i.e., $\lim_{t \to \infty} \boldsymbol{a}^{(t+0.5)} = \lim_{t \to \infty} \boldsymbol{a}^{(t)} = \boldsymbol{a}^*$

Proof of Theorem 9.4.1

By Lemma 9.4.1, we know that the set of ε-strong solutions of any concave pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ is equal to the set of ε-VE. Now, note that by the variational stability assumption, the set of weak solutions of $(\mathcal{X}^*, \boldsymbol{v})$ is non-empty. In addition, as $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ is a jointly λ-Lipschitz-smooth concave pseudo-games with jointly convex constraints, $(\mathcal{X}^*, \boldsymbol{v})$ is λ-Lipschitz continuous. Hence, the assumptions of Theorem 4.3.1 hold, giving us the result.

With this theorem in hand, so remarks are in order.

Remark 9.4.4 [Contributions to the literature]**.**

To the best of our knowledge, the above result is the first polynomial-time computation result for ε-VE, as well as ε-NE in variationally stable games. It is also the first and only existing non-asymptotic convergence analysis of the mirror extragradient learning dynamics.

We note that for the choice of kernel function $h \doteq \|\cdot\|^2$ (i.e., the euclidean square norm), while Huang and Zhang (2023) do not explicitly prove the above result, the result could be inferred from Huang and Zhang's Theorem 3.16 when taken in conjuction with Lemma 9.4.1. As such, for this particular setting, our contribution can be seen as identifying Lemma 9.4.1 to apply Huang and Zhang's result to pseudo-games.

Remark 9.4.5 [Local convergence to ε-VE]**.**

The above finite-time global convergence result to ε-VE, can be extended to a finite-time local convergence result to ε-VE by instead applying Theorem 4.3.2 with the assumption that the initial iterate of the mirror extragradient learning dynamics starts close enough to a local weak solution of $(\mathcal{X}^*, \boldsymbol{v})$. To the best of our knowledge this is the first finite-time local convergence result to ε-VE in pseudo-games, as well as ε-NE in games.

9.4.2 Merit Function Methods for GNE

We now turn our attention to methods with computational guarantees beyond variationally stable pseudo-games. In general concave pseudo-games, it is not possible for uncoupled learning dynamics to converge a VE (see, Theorem 1 of Hart and Mas-Colell (2003)). To remedy this non-convergence issue, we will in this section consider consider first-order *coupled* learning dynamics, or simply first-order learning dynamics. To derive, these first-order learning dynamics, we will define merit functions for VE and consider methods to minimize this merit functions.

Definition 9.4.6 [Merit functions for Pseudo-Games]**.**

Given a pseudo-game $(\mathcal{A}, \boldsymbol{h}\boldsymbol{u})$. A function $\Xi : \mathcal{X}^* \to \mathbb{R}$ is said to be a **merit function** for the set of VE of $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ iff

1. for all $\boldsymbol{a} \in \mathcal{X}^*$, $\Xi(\boldsymbol{a}) \geq 0$
2. for any $\boldsymbol{a}^* \in \mathcal{X}^*$, $\Xi(\boldsymbol{a}^*) = 0$ iff $\boldsymbol{a}^*$ is a VE.

Our formulations start with the exploitability, or the Nikaido-Isoda function (Nikaido and Isoda, 1955), as well as the related cumulative regret or Ky Fan function (Aubin, 2013) of a pseudo-game.

Definition 9.4.7 [Cumulative Regret and Exploitability]**.**

The **cumulative regret** (or **Ky Fan function**) $\psi : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$ between two action profiles $\boldsymbol{a}$ and $\boldsymbol{b}$ across all players is defined as:

$$\psi(\boldsymbol{a}, \boldsymbol{b}) \doteq \sum_{i \in [n]} \left[u_i(\boldsymbol{b}_i, \boldsymbol{a}_{-i}) - u_i(\boldsymbol{a}_i, \boldsymbol{a}_{-i})\right]$$

The **exploitability**, or the **Nikaido-Isoda *potential* function** (Nikaido and Isoda, 1955), $\varphi : \mathcal{A} \to \mathbb{R}$ of an action profile $\boldsymbol{a}$ is defined as

$$\varphi(\boldsymbol{a}) = \max_{\boldsymbol{b} \in \mathcal{X}^*} \sum_{i \in [n]} \left[u_i(\boldsymbol{b}_i, \boldsymbol{a}_{-i}) - u_i(\boldsymbol{a}_i, \boldsymbol{a}_{-i})\right] \quad .$$

A well-known is that any unexploitable action profile, (i.e., $\boldsymbol{a} \in \mathcal{X}(\boldsymbol{a})$ s.t. $\varphi(\boldsymbol{a}) = 0$) in a pseudo-game is a VE.

Lemma 9.4.2 [(Flam and Ruszczynski, 1994)]**.**

Given a pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$, for all $\boldsymbol{a} \in \mathcal{X}^*$, $\varphi(\boldsymbol{a}) \geq 0$, and any action profile $\boldsymbol{a} \in \mathcal{X}(\boldsymbol{a})$ with exploitability

$\varphi(\boldsymbol{a})$ is an $\varphi(\boldsymbol{a})$-NE. Further, any action profile $\boldsymbol{a}^{(*)} \in \mathcal{X}^*$ is a VE iff it achieves the lowerbound, i.e., $\varphi(\boldsymbol{a}^*) = 0$.

This lemma tells us that we can reformulate the VE computation problem as the optimization problem of minimizing exploitability, i.e., $\min_{\boldsymbol{a} \in \mathcal{X}^*} \varphi(\boldsymbol{a})$. Despite this reformulation of VE computation in terms of exploitability, no exploitability-minimization algorithms with convergence rate guarantees are known. This unexploitability (!) of exploitability may be due to the fact that it is not differentiable in general. The key insight that allows us to obtain convergence guarantees is that we treat the VE problem not as a minimization problem, but rather as a min-max optimization problem, namely:

$$\min_{\boldsymbol{a} \in \mathcal{X}^*} \varphi(\boldsymbol{a}) = \min_{\boldsymbol{a} \in \mathcal{X}^*} \max_{\boldsymbol{b} \in \mathcal{X}^*} \psi(\boldsymbol{a}, \boldsymbol{b})$$

This problem is well understood when ψ is a convex-concave objective function (Nemirovski, 2004; Korpelevich, 1976; Nedic and Ozdaglar, 2009; von Neumann, 1928). Furthermore, the cumulative regret ψ is indeed convex-concave, i.e., convex in $\boldsymbol{a}$ and concave in $\boldsymbol{b}$, in many pseudo-games of interest: e.g., two-player zero-sum, n-player pairwise zero-sum, and a large class of monotone and bilinear pseudo-games, as well as Cournot oligopoly games; for more details, see Section 2, (Flam and Ruszczynski, 1994).

Using the simple observation that every VE of a pseudo-game is the solution to a min-max optimization problem we introduce our first algorithm (EDA; Algorithm 8), an extragradient method (Korpelevich, 1976). The algorithm works by interleaving extragradient ascent and descent steps: at iteration t, given $\boldsymbol{a}^{(t)}$, it ascends on $\psi(\boldsymbol{a}^{(t)}, \cdot)$, thereby generating a better response $\boldsymbol{b}^{(t+1)}$, and then it descends on $\psi(\cdot, \boldsymbol{b}^{(t+1)})$, thereby decreasing exploitability. We combine several known results about the convergence of extragradient descent methods in min-max optimization problems to obtain the following convergence guarantees for EDA in pseudo-games.

Remark 9.4.6 [From EDA to Mirror Extragradient Descent Ascent]**.**

We note that one could more generally could consider a mirror extragradient descent ascent method which is equivalent to run the mirror extragradient algorithm (Algorithm 3) on the VI $(\mathcal{X}^* \times \mathcal{X}^*, (\nabla_{\boldsymbol{a}} \psi, -\nabla_{\boldsymbol{b}} \psi))$. The convergence results we provide in this section hold for this more general algorithm by simply applying Theorem 4.3.1 and other related theorem in the literature (e.g., Nemirovski (2004)). For simplicity, we chose to present them for this simpler algorithm.

Algorithm 8 Extragradient descent ascent (EDA)

Inputs: $\psi, \tau, \eta, \boldsymbol{a}^{(0)}, \boldsymbol{b}^{(0)}$

Outputs: $(\boldsymbol{a}^{(t)}, \boldsymbol{b}^{(t)}, \boldsymbol{a}^{(t+0.5)}, \boldsymbol{b}^{(t+0.5)})_t$

for $t = 0, \dots, \tau - 1$ **do**

$\quad \boldsymbol{a}^{(t+0.5)} = \Pi_{\mathcal{X}^*}\left[\boldsymbol{a}^{(t)} - \eta \nabla_{\boldsymbol{a}} \psi(\boldsymbol{a}^{(t)}, \boldsymbol{b}^{(t)})\right]$

$\quad \boldsymbol{b}^{(t+0.5)} = \Pi_{\mathcal{X}^*}\left[\boldsymbol{b}^{(t)} + \eta \nabla_{\boldsymbol{b}} \psi(\boldsymbol{a}^{(t)}, \boldsymbol{b}^{(t)})\right]$

$\quad \boldsymbol{a}^{(t+1)} = \Pi_{\mathcal{X}^*}\left[\boldsymbol{a}^{(t)} - \eta \nabla_{\boldsymbol{a}} \psi(\boldsymbol{a}^{(t+0.5)}, \boldsymbol{b}^{(t+0.5)})\right]$

$\quad \boldsymbol{b}^{(t+1)} = \Pi_{\mathcal{X}^*}\left[\boldsymbol{b}^{(t)} + \eta \nabla_{\boldsymbol{b}} \psi(\boldsymbol{a}^{(t+0.5)}, \boldsymbol{b}^{(t+0.5)})\right]$

return $(\boldsymbol{a}^{(t)}, \boldsymbol{b}^{(t)}, \boldsymbol{a}^{(t+0.5)}, \boldsymbol{b}^{(t+0.5)})_t$

Theorem 9.4.2 [Convergence of EDA]**.**

Consider a jointly λ-Lipschitz-smooth quasiconcave pseudo-game with jointly convex-constraints $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ with a convex-concave cumulative regret ψ. Suppose that EDA (Algorithm 8) is run with the cumulative regret ψ, the step size $\eta \leq \frac{1}{2n\lambda}$, time horizon $\tau \in \mathbb{N}$, initial iterates $\boldsymbol{a}^{(0)}, \boldsymbol{b}^{(0)} \in \mathcal{X}^*$ and that doing so generates the sequence of iterates $(\boldsymbol{a}^{(t)}, \boldsymbol{b}^{(t)}, \boldsymbol{a}^{(t+0.5)}, \boldsymbol{b}^{(t+0.5)})_t$. Let $\overline{\boldsymbol{a}^{(\tau)}} = {}^1\!/\!_\tau \sum_{t=1}^{T} \boldsymbol{a}^{(t)}$ and $\boldsymbol{a}^{(\tau')}_{\text{best}} \in \underset{\boldsymbol{a}^{(k+0.5)}:k=0,1,\dots,\tau'}{\arg\min} \|\boldsymbol{a}^{(k+0.5)} - \boldsymbol{a}^{(k)}\|$. Then, for any $\varepsilon \geq 0$, there exists some $\tau \in O\left(\frac{n\lambda}{\varepsilon}\right)$ and $\tau' \in O\left(\frac{n\lambda}{\varepsilon^2}\right)$ s.t. $\overline{\boldsymbol{a}^{(\tau)}}$ and $\boldsymbol{a}^{(\tau')}_{\text{best}}$ is a ε-VE.

Proof of Theorem 9.4.2

First, note that ψ is $2n\lambda$-Lipschitz-smooth since the pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ is jointly λ-Lipschitz-smooth, and ψ is the sum of n, differences of λ-Lipschitz-smooth functions. Additionally, since ψ is convex-concave, $(\mathcal{X}^* \times \mathcal{X}^*, (\nabla_{\boldsymbol{a}} \psi, -\nabla_{\boldsymbol{b}} \psi))$ is monotone. Further, note that any feasible action profile $\boldsymbol{a}^* \in \mathcal{X}^*$ is a ε-strong solution of $(\mathcal{X}^* \times \mathcal{X}^*, (\nabla_{\boldsymbol{a}} \psi, -\nabla_{\boldsymbol{b}} \psi))$ iff we have $\max_{\boldsymbol{b} \in \mathcal{X}^*} \psi(\boldsymbol{a}^*, \boldsymbol{b}) - \min_{\boldsymbol{a} \in \mathcal{X}^*} \max_{\boldsymbol{b} \in \mathcal{X}^*} \psi(\boldsymbol{a}, \boldsymbol{b}) \leq \varepsilon$ (see, Proposition 2.2 of Nemirovski (2004)). Since under the assumption of joint convexity a VE is guaranteed to exist (Theorem 9.2.2), it must be that

$\min_{\boldsymbol{a}\in\mathcal{X}^*}\max_{\boldsymbol{b}\in\mathcal{X}^*}\psi(\boldsymbol{a},\boldsymbol{b})=0$, meaning that we have:

$$\varepsilon \geq \max_{\boldsymbol{b}\in\mathcal{X}^*}\psi(\boldsymbol{a}^*,\boldsymbol{b}) - \underbrace{\min_{\boldsymbol{a}\in\mathcal{X}^*}\max_{\boldsymbol{b}\in\mathcal{X}^*}\psi(\boldsymbol{a},\boldsymbol{b})}_{=0}$$

$$= \max_{\boldsymbol{b}\in\mathcal{X}^*}\psi(\boldsymbol{a}^*,\boldsymbol{b}) = \varphi(\boldsymbol{a}^*)$$

As such, by Lemma 9.4.2, any ε-strong solution of $(\mathcal{X}^*\times\mathcal{X}^*,(\nabla_{\boldsymbol{a}}\psi,-\nabla_{\boldsymbol{b}}\psi))$ is a ε-VE.

Hence, since extragradient descent ascent (Algorithm 8) by Theorem 3.2 of Nemirovski (2004), we have that for all $\varepsilon\geq 0$, and $\tau\geq\frac{2n\lambda}{\varepsilon}$, $\overline{\boldsymbol{a}^{(T)}}$ is a ε-VE.

Similarly, by Theorem 4.3.1 we have that for all $\varepsilon\geq 0$, there exists $\tau'\in O(\frac{n\lambda}{\varepsilon})$ s.t. $\boldsymbol{a}^{(\tau')}_{\text{best}}$ is a ε-VE.

We note that EDA is an optimal algorithm for computing VE in pseudo-games with convex-concave cumulative regret.

Remark 9.4.7 [Optimality of EDA]**.**

Since the computational complexity of two-player zero-sum convex-concave games, a special case of pseudo-games with convex-concave cumulative regret, is $\Omega(1/\varepsilon)$, the iteration complexity of EDA in pseudo-games with convex-concave cumulative regret is optimal.

With the above theorem in hand, we also remark that the result can be extended beyond pseudo-games for which the cumulative regret is convex-concave. We present this less general result for the sake of simplicity, and as it answers an open question first posed by Flam and Ruszczynski (1994), which inquires whether if the exploitability can be minimized efficiently when ψ is convex-concave.

Remark 9.4.8 [EDA under Minty Condition]**.**

In general, going beyond pseudo-games for which ψ is convex-concave, we can consider pseudo-games for which the VI $(\mathcal{X}^*\times\mathcal{X}^*,(\nabla_{\boldsymbol{a}}\psi,-\nabla_{\boldsymbol{b}}\psi))$ satisfies the Minty condition. For such pseudo-games, by Theorem 4.3.1, it is still possible to obtain the polynomial-time computation result for ε-VE, i.e., for all $\varepsilon\geq 0$, there exists $\tau'\in O(\frac{n\lambda}{\varepsilon})$ s.t. $\boldsymbol{a}^{(\tau')}_{\text{best}}$ is a ε-VE.

Remark 9.4.9 [EDA in concave pseudo-games with jointly convex constraints]**.**

In general, for concave pseudo-games with jointly convex constraints, while the cumulative regret ψ is not convex-convex, it is guaranteed to be non-convex-concave. In this more general setting, as the φ is not differentiable, it is not possible to show convergence to a ε-stationary point of φ. Nevertheless, it is possible to show that extragradient descent ascent can compute a ε-stationary point of the Moreau envelope $\tilde{\varphi}$ of φ, defined as $\tilde{\varphi}(\boldsymbol{a}) \doteq \min_{\boldsymbol{b} \in \mathcal{X}^*} \varphi(\boldsymbol{b}) + \frac{1}{2n\lambda}\|\boldsymbol{a} - \boldsymbol{b}\|^2$ in $O(\frac{1}{\varepsilon^6})$ operations (see, for instance Mahdavinia et al. (2022)).

While Remark 9.4.9 suggests that it at least possible to minimize exploitability in general concave pseudo-games with jointly convex constraints, as ε-stationary points of $\tilde{\varphi}$ are not directly related to ε-stationary points of φ, it seems hard to relate this convergence result to the computation of ε-VE.

It is not immediately obvious how to design efficient algorithms that find stationary points of the exploitability, because the exploitability associated with the min-max characterization of pseudo-games is non-differentiable in general. Nonetheless, by exploiting the structure of cumulative regret, we regularize it to obtain a regularized exploitability function whose set of minima is once again equal to the set of VE. In particular, observe the following: if $\boldsymbol{a}^* \in \arg\min_{\boldsymbol{a} \in \mathcal{X}} \max_{\boldsymbol{b} \in \mathcal{X}} \psi(\boldsymbol{a}, \boldsymbol{b})$, then $\boldsymbol{a}^* \in \arg\max_{\boldsymbol{b} \in \mathcal{X}} \psi(\boldsymbol{a}^*, \boldsymbol{b})$. In other words, if $\boldsymbol{a}^*$ is a solution to the outer minimization problem, then it is likewise a solution to the inner maximization problem. As a result, we can penalize exploitability in proportion to the distance between $\boldsymbol{a}$ and $\boldsymbol{b}$, while still ensuring that this penalized exploitability is minimized at a VE. We thus propose to optimize the α-**regularized cumulative regret** $\psi_\alpha : \mathcal{A} \times \mathcal{A} \to \mathbb{R}$,

Definition 9.4.8 [Regularized Cumulative Regret and Regularized Exploitability]**.**
Let $\alpha \geq 0$ be a regularization parameter.

The α-**regularized cumulative regret** of any two action profiles $\boldsymbol{a}, \boldsymbol{b} \in \mathcal{A}$ is defined as:

$$\psi_\alpha(\boldsymbol{a}, \boldsymbol{b}) \doteq \psi(\boldsymbol{a}, \boldsymbol{b}) - \frac{\alpha}{2} \|\boldsymbol{a} - \boldsymbol{b}\|_2^2$$

The α-**regularized exploitability** $\varphi_\alpha : \mathcal{A} \to \mathbb{R}$ of any action is defined as:

$$\varphi_\alpha(\boldsymbol{a}) \doteq \max_{\boldsymbol{b} \in \mathcal{X}} \psi_\alpha(\boldsymbol{a}, \boldsymbol{b})$$

Von Heusinger and Kanzow show that an action profile $\boldsymbol{a}^*$ has no α-regularized-exploitability, i.e., $\varphi_\alpha(\boldsymbol{a}^*) = 0$, iff $\boldsymbol{a}^*$ is a VE, for all $\alpha \geq 0$ (Theorem 3.3 (Von Heusinger and Kanzow, 2009)). With this observation in

Algorithm 9 Regularized Extragradient descent ascent (REDA)

Inputs: $\psi_\alpha, \tau, \eta, \boldsymbol{a}^{(0)}, \boldsymbol{b}^{(0)}$

Outputs: $(\boldsymbol{a}^{(t)}, \boldsymbol{b}^{(t)}, \boldsymbol{a}^{(t+0.5)}, \boldsymbol{b}^{(t+0.5)})_t$

1: **for** $t = 0, \dots, \tau - 1$ **do**

2: $\quad \boldsymbol{a}^{(t+0.5)} = \Pi_{\mathcal{X}^*}\left[\boldsymbol{a}^{(t)} - \eta \nabla_{\boldsymbol{a}} \psi_\alpha(\boldsymbol{a}^{(t)}, \boldsymbol{b}^{(t)})\right]$

3: $\quad \boldsymbol{b}^{(t+0.5)} = \Pi_{\mathcal{X}^*}\left[\boldsymbol{b}^{(t)} + \eta \nabla_{\boldsymbol{b}} \psi_\alpha(\boldsymbol{a}^{(t)}, \boldsymbol{b}^{(t)})\right]$

4: $\quad \boldsymbol{a}^{(t+1)} = \Pi_{\mathcal{X}^*}\left[\boldsymbol{a}^{(t)} - \eta \nabla_{\boldsymbol{a}} \psi_\alpha(\boldsymbol{a}^{(t+0.5)}, \boldsymbol{b}^{(t+0.5)})\right]$

5: $\quad \boldsymbol{b}^{(t+1)} = \Pi_{\mathcal{X}^*}\left[\boldsymbol{b}^{(t)} + \eta \nabla_{\boldsymbol{b}} \psi_\alpha(\boldsymbol{a}^{(t+0.5)}, \boldsymbol{b}^{(t+0.5)})\right]$

6: **return** $(\boldsymbol{a}^{(t)}, \boldsymbol{b}^{(t)}, \boldsymbol{a}^{(t+0.5)}, \boldsymbol{b}^{(t+0.5)})_t$

hand, we can then try to minimize the regularized exploitability by running extragradient descent ascent on ψ_α, rather than ψ, which gives us the regularized extragradient descent ascent algorithm (Algorithm 9).

Theorem 9.4.3 [Convergence of REDA]**.**

Consider a jointly λ-Lipschitz-smooth concave pseudo-game with jointly convex-constraints $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$. Let $\alpha > 0$ be some regularization parameter, and ψ_α be the α-regularized cumulative regret associated with $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$.

Suppose that REDA (Algorithm 9) is run with is run with the regularized cumulative regret ψ_α, the step size $\eta \leq \min\{\frac{1}{75\alpha(2n\lambda+\alpha)}, \frac{1}{4(2n\lambda+\alpha)}\}$, time horizon $\tau \in \mathbb{N}$, initial iterates $\boldsymbol{a}^{(0)}, \boldsymbol{b}^{(0)} \in \mathcal{X}^*$ and that doing so generates the sequence of iterates $(\boldsymbol{a}^{(t)}, \boldsymbol{b}^{(t)}, \boldsymbol{a}^{(t+0.5)}, \boldsymbol{b}^{(t+0.5)})_t$.

Let $\boldsymbol{a}^{(\tau)}_{\text{best}} \in \arg\min_{\boldsymbol{a}^{(k)}:k=0,\dots,\tau-1} \max_{\boldsymbol{a}\in\mathcal{X}^*} \langle \nabla \varphi_\alpha(\boldsymbol{a}^{(k)}), \boldsymbol{a}^{(k)} - \boldsymbol{a}\rangle$, then, for any $\varepsilon \geq 0$, there exists some choice of time horizon $\tau \in O(\frac{1}{\varepsilon^2})$, $\boldsymbol{a}^{(\tau)}_{\text{best}}$ is a ε-stationary point of φ_α, i.e., $\max_{\boldsymbol{a}\in\mathcal{X}^*} \langle \nabla \varphi_\alpha(\boldsymbol{a}^{(\tau)}_{\text{best}}), \boldsymbol{a}^{(\tau)}_{\text{best}} - \boldsymbol{a}\rangle \leq \varepsilon$.

Proof of Theorem 9.4.3

The result follows from an application of Theorem 4.2 Mahdavinia et al. (2022) by noting the following. First, note that Theorem 4.2 is derived for the min-max optimization problems where the minimization is unconstrained, which in our application corresponds to the optimization problem $\min_{\boldsymbol{a}\in\mathbb{R}^{n\times m}} \max_{\boldsymbol{b}\in\mathcal{X}^*} \psi(\boldsymbol{a}, \boldsymbol{b})$. Nevertheless, as Mahdavinia et al. remark at the end of their related works section, their proof directly generalizes to the constrained setting, with a definition of a ε-stationary point of φ_α given by $\max_{\boldsymbol{x}\in\mathcal{X}} \langle \nabla \varphi_\alpha(\boldsymbol{a}^{(\tau)}_{\text{best}}), \boldsymbol{a}^{(\tau)}_{\text{best}} - \boldsymbol{a} \rangle \leq \varepsilon$.

Second, for all $\boldsymbol{a} \in \mathcal{A}$, $\boldsymbol{b} \mapsto \psi_\alpha(\boldsymbol{a}, \boldsymbol{b})$ is α is α-strongly-concave since it is the sum of a concave function and α-strongly-concave function.

Finally, note that ψ_α is $(2n\lambda + \alpha)$-Lipschitz-smooth since it the sum of n differences of λ functions ψ, and $\frac{\alpha}{2} \|\boldsymbol{a} - \boldsymbol{b}\|_2^2$ which is α-Lipschitz-smooth. Hence, setting the step size so that $\eta \leq \min\{\frac{1}{75\alpha(2n\lambda+\alpha)}, \frac{1}{4(2n\lambda+\alpha)}\}$, the antecedant of Theorem 4.2 of Mahdavinia et al. (2022) is satisfied, and the result follows.

With this convergence result in hand, one might wonder under what conditions ε-stationary points of ψ_α coincide with ε-VE.

Remark 9.4.10 [When is a ε-stationary point of φ_α, a ε-VE]**.**

Von Heusinger and Kanzow (2009) show in Theorem 3.6 that when the pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ satisfies the following strict monotonicity-like condition for all $\boldsymbol{a}, \boldsymbol{b}$ s.t. $\boldsymbol{a} \neq \boldsymbol{b}$:

$$\sum_{i\in[n]} \left\langle \nabla u_i(\boldsymbol{b}_i, \boldsymbol{a}_{-i}) - \nabla u_i(\boldsymbol{a}), \boldsymbol{a} - \boldsymbol{b} \right\rangle > 0$$

then ε-stationary points of ψ_α coincide with ε-VE of $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$.[3]

This condition can be translated to a more intuitive condition on $\boldsymbol{u}$ by noting that it is satisfied when for all players $i \in [n]$, $\boldsymbol{a} \mapsto u_i(\boldsymbol{a})$ is strictly convex.

Remark 9.4.11 [Extending Convergence to Quasiconcave Pseudo-Games]**.**

We note that while the for all $\boldsymbol{a} \in \mathcal{A}$, $\boldsymbol{b} \mapsto \psi(\boldsymbol{a}, \boldsymbol{b})$ is not concave in quasiconcave pseudo-games, under the assumption that the pseudo-game is jointly λ-Lipschitz-smooth, $\boldsymbol{b} \mapsto \psi(\boldsymbol{a}, \boldsymbol{b})$ is λ-weakly-concave. Hence, choosing $\alpha > \lambda$, we can ensure that $\boldsymbol{b} \mapsto \psi_\alpha(\boldsymbol{a}, \boldsymbol{b})$ is strongly-concave, hence allowing us to extend Theorem 9.4.3 to jointly λ-Lipschitz-smooth quasiconcave pseudo-games with jointly convex constraints.

Unfortunately, beyond quasiconcave pseudo-games with jointly convex constraints, a VE is not guaranteed to exist, and as such minimizing the (regularized) exploitability is no longer a sensible approach. Never-

[3] Von Heusinger and Kanzow (2009) show the result for 0-stationary points and 0-VE, however, their proof directly generalizes to ε-stationary points by replacing the 0s in their proof with ε.

theless, it is possible to show the existence of a weaker solution concept, namely the first-order variational equilibrium in smooth pseudo-games. As such, we next turn our attention to the computation of first-order variational equilibrium.

9.5 Local Solution Concepts and Existence

9.5.1 First-Order and Local Generalized Nash and Variational Equilibrium

We will now turn our attention to solving non-concave games. As we have previously mentioned, the computation of a ε-NE even in single player quasiconcave games (i.e., quasiconcave optimization) is known to be NP-hard (Vavasis, 1995). As such, to obtain any computational results for efficient algorithms even in quasiconcave game we have to set our sights on the computation of an alternative local solution concept.

Further, as many pseudo-games that arise in modern machine learning applications are not quasiconcave, and GNE are not guaranteed to exist beyond quasiconcave games, this begs the question of how far beyond quasiconcave games can the theory of games be extended, and what solution concepts are appropriate for such pseudo-games. Two intuitive solution concepts which are weaker than GNE are the first-order generalized Nash equilibrium and the local generalized Nash equilibrium.

Definition 9.5.1 [First-Order GNE]**.**
Given $\varepsilon \geq 0$, A ε-**first-order generalized Nash equilibrium** (ε-**first-order GNE**) is an action profile $\boldsymbol{a}^* \in \mathcal{X}(\boldsymbol{a}^*)$ s.t. for all players $i \in [n]$ and $\boldsymbol{a}_i \in \mathcal{X}_i(\boldsymbol{a}^*_{-i})$,

$$\left\langle \partial_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}_i - \boldsymbol{a}_i^* \right\rangle \leq \varepsilon$$

for some $\partial_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*) \in \mathcal{D}_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*)$.

A 0-first-order GNE is simply called a first-order generalized Nash equilibrium (first-order GNE).

Remark 9.5.1 [Interpretation of first-order GNE]**.**
First-order GNE can be interpreted as the GNE of the pseudo-game with "linearized" payoffs around the GNE. More precisely, for all $i \in [n]$, let $\ell_i^{\boldsymbol{a}^*}[u_i](\boldsymbol{a}_i) \doteq u_i(\boldsymbol{a}^*) + \left\langle \partial_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}_i - \boldsymbol{a}_i^* \right\rangle$ be the linearization operator around $\boldsymbol{a}^*$ s.t. $\ell_i^{\boldsymbol{a}^*}[u_i](\boldsymbol{a}_i)$ provides a first-order Taylor expansion approximation for $u_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i})$.

Recall that a GNE is an action profile $\boldsymbol{a}^* \in \mathcal{X}(\boldsymbol{a}^*)$ s.t. for all $i \in [n]$ and $\boldsymbol{a}_i \in \mathcal{X}_i(\boldsymbol{a}^*_{-i})$:

$$u_i(\boldsymbol{a}^*) \geq u_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i}) \ .$$

Now, suppose that we replace the payoffs $\boldsymbol{u}$ in the original pseudo-game with the linearized payoffs around $\boldsymbol{a}^*$, then for all $i \in [n]$ and $\boldsymbol{a}_i \in \mathcal{X}_i(\boldsymbol{a}^*_{-i})$ the above definition reduces to:

$$\begin{aligned} & \ell_i^{\boldsymbol{a}^*}[u_i](\boldsymbol{a}_i^*) \geq \ell_i^{\boldsymbol{a}^*}[u_i](\boldsymbol{a}_i) \\ \iff & u_i(\boldsymbol{a}^*) \geq u_i(\boldsymbol{a}^*) + \left\langle \partial_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}_i - \boldsymbol{a}_i^* \right\rangle \\ \iff & 0 \geq \left\langle \partial_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}_i - \boldsymbol{a}_i^* \right\rangle \end{aligned}$$

This interpretation is key in the analysis of many general equilibrium models used in modern macroeconomics, as in most if not all models used in practice the cumulative expected utility of the consumers are linearized before solving for a solution of the model. As such, the first-order GNE provides a theoretical framework to understand this trick used in practice.

We can similarly define a first-order analog of VE, which is a refinement of the first-order GNE.

Definition 9.5.2 [First-Order VE]**.**
Given $\varepsilon \geq 0$, A ε-**first-order variational equilibrium** (ε-**first-order VE**) is an action profile $\boldsymbol{a}^* \in \mathcal{X}(\boldsymbol{a}^*)$ s.t. for all $\boldsymbol{a} \in \mathcal{X}^*$:

$$\sum_{i \in [n]} \left\langle \partial_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}_i - \boldsymbol{a}_i^* \right\rangle \leq \varepsilon$$

for some $\partial_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*) \in \mathcal{D}_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*)$.

A 0-first-order VE is simply called a first-order variational equilibrium (first-order VE).

Note that the set of first-order VE of any pseudo-game is a subset of the set of local GNE. An alternative to the first-order GNE is the local GNE.

Definition 9.5.3 [Local GNE]**.**
Given a **regret parameter** $\varepsilon \geq 0$, and a **locality parameter** $\delta \geq 0$, a (ε, δ)**-local generalized Nash equilibrium** ((ε, δ)**-local GNE**) is an action profile $\boldsymbol{a}^* \in \mathcal{A}$ s.t. for all players $i \in [n]$ and $\boldsymbol{a}_i \in \mathcal{A}_i \cap \mathcal{B}_\delta[\boldsymbol{a}_i^*]$:

$$u_i(\boldsymbol{a}^*) \geq u_i(\boldsymbol{a}_i, \boldsymbol{a}_{-i}^*) - \varepsilon \ .$$

For any $\delta > 0$, a $(0, \delta)$-local GNE is simply called a local generalized Nash equilibrium (local GNE).

Similarly, we can define a local analog of VE.

Definition 9.5.4 [Local VE]**.**

Given a **regret parameter** $\varepsilon \geq 0$, and a **locality parameter** $\delta \geq 0$, a (ε, δ)**-local variational equilibrium** ((ε, δ)**-local VE**) is an action profile $\boldsymbol{a}^* \in \mathcal{A}$ s.t. for all $\boldsymbol{a} \in \mathcal{X}^* \cap \mathcal{B}_\delta[\boldsymbol{a}^*]$:

$$\sum_{i \in [n]} u_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i}) - u_i(\boldsymbol{a}^*) \leq \varepsilon \ .$$

For any $\delta > 0$, a $(0, \delta)$-local VE is called a local variational equilibrium (local VE).

Similarly, to their global variants, the set of local VE is a subset of the set of local GNE.

Remark 9.5.2 [First-Order NE and local NE]**.**

For games, the definitions of first-order GNE and first-order VE are equivalent in which case these solution concepts are simply called **first-order Nash equilibrium** (**first-order NE**). Similarly, in games, the definitions of local GNE and local VE are also equivalent in which case these solution concepts are simply called **local Nash equilibrium** (**local NE**).

9.5.2 Smooth Pseudo-Games

Unfortunately, it is not possible to guarantee the existence of a local GNE even in very simply non-concave games.

Example 9.5.1 [Exact local GNE non-existence]**.**

Consider the two player zero-sum game $(2, \mathcal{A}, \boldsymbol{u})$ where $\mathcal{A} \doteq [-1, 1] \times [-1, 1]$, and $u_1(a_1, a_2) = -u_2(a_1, a_2) = (a_1 - a_2)^2$. At a local Nash equilibrium $(a_1^*, a_2^*) \in \mathcal{A}$, player 1 would play $a_1 \neq a_2$, while player 2 would play $a_1 = a_2$. Hence, an exact local NE cannot exist.

Nevertheless for any $\varepsilon, \delta \geq 0$ s.t. $\varepsilon \geq 2\delta^2$, any action profile $(a_1^*, a_2^*) \in \mathcal{A}$ s.t. $a_1^* = a_2^*$ is a (ε, δ)-local NE since:

$$\begin{aligned} &\max_{a_1 \in [-1,1] \cap \mathcal{B}_\delta[a_1^*]} u_1(a_1, a_2^*) && \max_{a_2 \in [-1,1] \cap \mathcal{B}_\delta[a_2^*]} u_2(a_1^*, a_2) \\ &= \max_{a_1 \in [-1,1] \cap \mathcal{B}_\delta[a_2^*]} (a_1 - a_2^*)^2 && \max_{a_2 \in [-1,1] \cap \mathcal{B}_\delta[a_2^*]} -(a_2^* - a_2)^2 \\ &\leq \delta^2 && = 0 \end{aligned}$$

In contrast, first-order GNE can be shown to exist in smooth pseudo-games.

Definition 9.5.5.

A **smooth pseudo-game** is a pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ where for all players $i \in [n]$:

[Continuous payoffs] $\nabla_{\boldsymbol{a}_i} u_i$ is continuous;

[Convex constraints] $\mathcal{X}_{-i}$ is continuous, non-empty-, compact-, and convex-valued;

[Convex action space] $\mathcal{A}_i$ is non-empty, compact, and convex.

Theorem 9.5.1 [Existence of first-order GNE]**.**

A first-order GNE exists in any smooth pseudo-game.

Proof of Theorem 9.5.1

Consider the first-order best-response correspondence $\mathcal{FOBR}(\boldsymbol{a}) \doteq \times_{i\in[n]} \arg\min_{\boldsymbol{a}'_i \in \mathcal{X}_i(\boldsymbol{a}_{-i})} \left\{\left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}), \boldsymbol{a}'_i \right\rangle\right\}$. Note that at any fixed point $\boldsymbol{a}^*$ s.t. $\boldsymbol{a}^* \in \mathcal{FOBR}(\boldsymbol{a}^*)$, we have for all players $i \in [n]$:

$$\boldsymbol{a}_i^* \in \arg\min_{\boldsymbol{a}'_i \in \mathcal{X}_i(\boldsymbol{a}^*_{-i})} \left\{\left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}'_i \right\rangle\right\}$$

Or equivalently, for all players $i \in [n]$ and $\boldsymbol{a}_i \in \mathcal{X}_i(\boldsymbol{a}^*_{-i})$:

$$\left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}_i^* \right\rangle \leq \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}_i \right\rangle$$
$$\iff \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}_i^* - \boldsymbol{a}_i \right\rangle \leq 0$$

Hence, $\boldsymbol{a}^*$ is first-order GNE.

Now, note that by the Berge's maximum theorem Berge (1997), in smooth games, the first-order best-response correspondence $\mathcal{FOBR}$ is upper hemicontinuous, and non-empty-, compact-, convex-valued. Hence, by the Kakutani-Glicksberg fixed point theorem (Theorem 2.4.1) a fixed point of $\mathcal{FOBR}$, and hence a first-order GNE is guaranteed to exist.

A similar existence result can be extended to VE under the additional assumption that the pseudo-game has jointly convex constraints.

Theorem 9.5.2 [Existence of first-order VE]**.**

A first-order VE exists in any smooth pseudo-game with jointly convex constraints.

Proof of Theorem 9.5.2

Consider the variational first-order best-response correspondence $\mathcal{VFOBR}(\boldsymbol{a}) \doteq \arg\min_{\boldsymbol{a}' \in \mathcal{X}^*} \left\{ \sum_{i \in [n]} \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}), \boldsymbol{a}'_i \right\rangle \right\}$. Note that at any fixed point $\boldsymbol{a}^*$ s.t. $\boldsymbol{a}^* \in \mathcal{VFOBR}(\boldsymbol{a}^*)$, we have for all players $i \in [n]$:

$$\boldsymbol{a}^* \in \arg\min_{\boldsymbol{a}'_i \in \mathcal{X}^*} \left\{ \sum_{i \in [n]} \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}'_i \right\rangle \right\}$$

Or equivalently, for all players $i \in [n]$ and $\boldsymbol{a} \in \mathcal{X}^*$:

$$\sum_{i \in [n]} \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}^*_i \right\rangle \leq \sum_{i \in [n]} \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}_i \right\rangle$$

$$\iff \sum_{i \in [n]} \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}^*_i - \boldsymbol{a}_i \right\rangle \leq 0$$

Hence, $\boldsymbol{a}^*$ is first-order VE.

Now, note that by the Berge's maximum theorem Berge (1997), in smooth games with jointly convex constraints, the variational first-order best-response correspondence $\mathcal{VFOBR}$ is upper hemicontinuous, and non-empty-, compact-, convex-valued. Hence, by the Kakutani-Glicksberg fixed point theorem (Theorem 2.4.1) a fixed point of $\mathcal{VFOBR}$, and hence a first-order VE is guaranteed to exist.

9.5.3 First-Order and Local Equilibrium Equivalence

While a local GNE is not guaranteed to exist in general, in λ-Lipschitz-smooth pseudo-games, it is possible to show that any ε-first-order GNE is a $(\varepsilon + \frac{\lambda\delta^2}{2}, \delta)$-local GNE, hence guaranteeing the existence of a $(\frac{\lambda\delta^2}{2}, \delta)$-local GNE for any choice of $\delta \geq 0$ in Lipschitz-smooth pseudo-games by the existence of first-order GNE.

Lemma 9.5.1 [First-Order and Local GNE Equivalence]**.**

Consider a λ-Lipschitz-smooth game, and some $\varepsilon, \delta \geq 0$. Then, any ε-first-order GNE (resp. VE) is a $(\varepsilon + \frac{\lambda\delta^2}{2}, \delta)$-local GNE (resp. -local VE).

Proof of Lemma 9.5.1

Given a ε-first-order GNE $\boldsymbol{a}^* \in \mathcal{A}$, by the weak-concavity property of Lipschitz-smooth functions, we have for all players $i \in [n]$ and $\boldsymbol{a}_i \in \mathcal{X}_i(\boldsymbol{a}^*_{-i})$:

$$
\begin{aligned}
u_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i}) &\leq u_i(\boldsymbol{a}^*) + \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}_i - \boldsymbol{a}^*_i \right\rangle + \frac{\lambda}{2} \left\| \boldsymbol{a}_i - \boldsymbol{a}^*_i \right\|^2 \\
u_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i}) &\leq u_i(\boldsymbol{a}^*) + \varepsilon + \frac{\lambda}{2} \left\| \boldsymbol{a}_i - \boldsymbol{a}^*_i \right\|^2 \\
u_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i}) - u_i(\boldsymbol{a}^*) &\leq \varepsilon + \frac{\lambda}{2} \left\| \boldsymbol{a}_i - \boldsymbol{a}^*_i \right\|^2
\end{aligned}
$$

where the penultimate line follows from the definition of a ε-first-order GNE. Re-organizing expressions, for all players $i \in [n]$, $\boldsymbol{a} \in \mathcal{X}_i(\boldsymbol{a}_{-i}) \cap \mathcal{B}_\delta[\boldsymbol{a}^*_i]$:

$$
u_i(\boldsymbol{a}_i, \boldsymbol{a}^*_{-i}) - u_i(\boldsymbol{a}^*) \leq \varepsilon + \frac{\lambda \delta^2}{2} \tag{9.1}
$$

That is, $\boldsymbol{a}^*$ is a $(\varepsilon + \frac{\lambda\delta^2}{2}, \delta)$-local Nash equilibrium.

The proof follows similarly for the case of first-order VE, and local VE.

9.6 Computation of First-Order and Local GNE

9.6.1 First-Order Variational Equilibrium and Variational Inequalities

We first present a characterization of first-order VE in pseudo-games in terms of strong solutions of VIs. The following intuitive generalization of Lemma 9.4.1 is guaranteed to hold for ε-first-order VE in all pseudo-games. We present the result for smooth pseudo-games as the computational results we will derive will require the payoff functions of the players to be differentiable. Nevertheless, the result directly generalizes to all pseudo-games by replacing the VI in the statement of Lemma 9.6.1 with the one presented in Remark 9.4.1.

Lemma 9.6.1 [SVI = VE in Smooth Pseudo-Games]**.**
The set of ε-first-order VE of any smooth pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ is equal to the set of ε-strong solutions $\mathcal{SVI}_\varepsilon(\mathcal{X}^*, \boldsymbol{v})$ of the VI $(\mathcal{X}^*, \boldsymbol{v})$.

Proof of Lemma 9.6.1

($\implies$) : Let $\boldsymbol{a}^* \in \mathcal{X}^*$ be a first-order VE of $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$, then for all players $i \in [n]$ and $\boldsymbol{a} \in \mathcal{X}^*$, we have:

$$\varepsilon \geq \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}_i - \boldsymbol{a}_i^* \right\rangle$$

Summing up the above inequality across all players $i \in [n]$, we have for all feasible action profiles $\boldsymbol{a} \in \mathcal{X}^*$:

$$\varepsilon \geq \sum_{i \in [n]} \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}_i - \boldsymbol{a}_i^* \right\rangle$$

$$\varepsilon \geq \langle -\boldsymbol{v}(\boldsymbol{a}^*), \boldsymbol{a} - \boldsymbol{a}^* \rangle$$

$$\varepsilon \geq \langle \boldsymbol{v}(\boldsymbol{a}^*), \boldsymbol{a}^* - \boldsymbol{a} \rangle$$

Hence, $\boldsymbol{a}^*$ is a ε-strong solution of the VI $(\mathcal{X}^*, \boldsymbol{v})$.

($\impliedby$) : Let $\boldsymbol{a}^* \in \mathcal{X}^*$ be a ε-strong solutions of the VI $(\mathcal{X}^*, \boldsymbol{v})$, then we have for all $\boldsymbol{a} \in \mathcal{X}^*$:

$$\varepsilon \geq \langle \boldsymbol{v}(\boldsymbol{a}^*), \boldsymbol{a}^* - \boldsymbol{a} \rangle$$

Letting for all feasible action profiles $\boldsymbol{a}' \in \mathcal{X}^*$, and players $i \in [n]$,

$$\boldsymbol{a}_k \doteq \begin{cases} \boldsymbol{a}'_i & \text{if } k = i \\ \boldsymbol{a}^*_k & \text{otherwise} \end{cases},$$

we have for all $\boldsymbol{a}' \in \mathcal{X}^*$:

$$\begin{aligned} \varepsilon &\geq \left\langle -\nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}^*_i - \boldsymbol{a}_i \right\rangle \\ &\geq \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}^*), \boldsymbol{a}_i - \boldsymbol{a}^*_i \right\rangle \end{aligned}$$

Hence, $\boldsymbol{a}^*$ is a ε-first-order VE of $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$.

Since the set of first-order VE is equal to the set of first-order NE in games, and $\mathcal{X}^* = \mathcal{A}$, we also have the following corollary of Lemma 9.6.1.

Corollary 9.6.1.

The set of ε-NE of any smooth game $(\mathcal{A}, \boldsymbol{u})$ is equal to the set of ε-strong solutions of the VI $(\mathcal{A}, \boldsymbol{v})$.

9.6.2 Uncoupled Learning Dynamics First-Order GNE

With a characterization of first-order VE in terms of strong solutions of a VI in hand, we now turn our attention to the computation of first-order VE. Beyond variationally stable concave pseudo-games, first-order methods are not guaranteed to converge to VE and thus to GNE.[4] Thus, a very natural question to ask is what convergence guarantees can be obtained in general variationally stable pseudo-games which are not necessarily concave.

The class of variationally stable pseudo-games contains the class of quasimonotone (resp. monotone // pseudomonotone) smooth pseudo-games with jointly convex constraints (see, for instance Huang and Zhang (2023)). Unfortunately, the class of quasimonotone games does not take us further out than the class of quasiconcave games as the following remark describes.

[4]See, for instance Example 9.6.1 and observe that first-order methods can converge to $(0, 0)$, which is not a VE.

Remark 9.6.1 [Quasimonotone Pseudo-Games are Quasiconcave]**.**

We note that any quasimonotone game is quasiconcave. To see this, first recall that a function is quasiconvex iff its subdifferential is quasimonotone (see, for instance, Theorem 4.1 of (Aussel et al., 1994)).

Now, setting for all $k \neq i \in [n]$, $\boldsymbol{b}_k \doteq \boldsymbol{a}_k$, the quasimonotonicity condition implies: for all i, $\boldsymbol{a}_i, \boldsymbol{b}_i \in \mathcal{A}_i$,

$$\left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{a}), \boldsymbol{a}_i - \boldsymbol{b}_i \right\rangle < 0 \implies \left\langle \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{b}_i, \boldsymbol{a}_{-i}), \boldsymbol{a}_i - \boldsymbol{b}_i \right\rangle \leq 0$$

Hence for all $i \in [n]$, $\boldsymbol{a}_{-i} \in \mathcal{A}_{-i}$, the mapping $\boldsymbol{b}_i \mapsto \nabla_{\boldsymbol{a}_i} u_i(\boldsymbol{b}_i, \boldsymbol{a}_{-i})$ is quasimonotone, implying that $\boldsymbol{b}_i \mapsto u_i(\boldsymbol{b}_i, \boldsymbol{a}_{-i})$ is quasiconcave.

Nevertheless, while the class of quasimonotone pseudo-games contains the class quasiconcave pseudo-games, in general variationally stable games are not necessarily quasiconcave as shown by the following example.

Example 9.6.1 [Variationally Stable game is not quasiconcave]**.**

Consider the single player game $(1, \mathcal{A}, \boldsymbol{u})$, where $\mathcal{A}_1 = [-1, 1]^2$ and $u_1 \doteq (\boldsymbol{a}_1) \doteq 1/3 a_{11}^3 + 1/3 a_{12}^3$, equivalently stated as the following single objective optimization problem:

$$\max_{a_{11}, a_{12} \in [-1,1]} u_i(\boldsymbol{a}_1) \doteq \frac{1}{3} a_{11}^3 + \frac{1}{3} a_{12}^3$$

For this game, we have $\nabla u_1(\boldsymbol{a}_1) \doteq (a_{11}^2, a_{12}^2)$. Now, notice that this game is variationally stable since for $\boldsymbol{a}_1^* \doteq (1, 1)$, we have for all $\boldsymbol{b}_1 \in [-1, 1]$, $\nabla u_1(\boldsymbol{b}_1) \geq 0$, and hence $\langle \nabla u_1(\boldsymbol{b}), \boldsymbol{a}_i^* - \boldsymbol{b}_i \rangle \geq 0$. However, notice that u_1 is not quasiconcave, since $u_1(1/2(1, 0) + 1/2(1/2, 1)) = u_1(3/4, 1/2) = \frac{35}{192} \approx 0.182$, but $u_1(1, 0) = \frac{1}{3} \approx 0.333$ and $u_1(1/2, 1) = \frac{3}{8} \approx 0.375$, meaning that $u_1(1/2(1, 0) + 1/2(1/2, 1)) < \min\{u_1(1, 0), u_1(1/2, 1)\}$.

This suggests that the class of variational stable pseudo-games is an interesting and broad enough class of non-quasiconcave pseudo-games to study. Further, by Lemma 9.6.1, since variational stability ensures that the VI $(\mathcal{X}^*, \boldsymbol{v})$ satisfies the Minty condition, it is also a sufficient condition to ensure the convergence of first-order methods. In particular, recall that by applying the mirror extragradient method to solve the VI $(\mathcal{X}^*, \boldsymbol{v})$ we had obtained the mirror extragradient learning dynamics (Algorithm 7). Hence, applying Theorem 4.3.1 in conjuction with Lemma 9.6.1, we obtain the following convergence theorem for the mirror extragradient learning dynamics in variationally stable pseudo-games with jointly convex constraints.

Theorem 9.6.1 [Convergence of Mirror extragradient Learning Dynamics]**.**

Let $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ be a variationally stable and λ-Lipschitz-smooth pseudo-game with jointly convex constraints, and h a 1-strongly-convex and κ-Lipschitz-smooth kernel function. Consider the mirror extragradient learning dynamics (Algorithm 7) run with the pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$, the kernel function h, a step size $\eta \in \left(0, \frac{1}{\sqrt{2}\lambda}\right]$, for any time horizon $\tau \in \mathbb{N}$, and outputs $\{\boldsymbol{x}^{(t+0.5)}, \boldsymbol{x}^{(t+1)}\}_t$. Let $\boldsymbol{a}^{(\tau)}_{\text{best}} \in \arg\min_{\boldsymbol{x}^{(k+0.5)}:k=0,\ldots,\tau} \text{div}_h(\boldsymbol{a}^{(k+0.5)}, \boldsymbol{a}^{(k)})$. Then, for some choice of $\tau \in O(\frac{1}{\varepsilon^2})$, $\boldsymbol{a}^{(\tau)}_{\text{best}}$ is a ε-first-order VE of $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$. In addition, the iterates generated asymptotically converge to some first-order VE $\boldsymbol{a}^* \in \mathcal{X}^*$ of $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$, i.e., $\lim_{t\to\infty} \boldsymbol{a}^{(t+0.5)} = \lim_{t\to\infty} \boldsymbol{a}^{(t)} = \boldsymbol{a}^*$

Proof of Theorem 9.6.1

By Lemma 9.6.1, we know that the set of ε-strong solutions of any concave pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ is equal to the set of ε-VE. Now, note that by the variational stability assumption, the set of weak solutions of $(\mathcal{X}^*, \boldsymbol{v})$ is non-empty. In addition, as $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ is a jointly λ-Lipschitz-smooth concave pseudo-games with jointly convex constraints, $(\mathcal{X}^*, \boldsymbol{v})$ is λ-Lipschitz continuous. Hence, the assumptions of Theorem 4.3.1 hold, giving us the result.

With this theorem in hand, we make two remarks on its implications on the local VE (or GNE), and the local convergence properties of the algorithm.

Remark 9.6.2 [Computation of Local GNE/VE]**.**

Applying Lemma 9.5.1, under the assumptions of Theorem 9.6.1, we can show that for all $\varepsilon, \delta \geq 0$ s.t. $\varepsilon \geq \frac{\lambda\delta^2}{2}$, there exists some choice of $\tau \in O(\frac{1}{\varepsilon^2})$, $\boldsymbol{a}^{(\tau)}_{\text{best}}$ is a (ε, δ)-local VE of $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$. Choices of (ε, δ) s.t. $\varepsilon \geq \frac{\lambda\delta^2}{2}$ have previously been known under the name of local parameter regimes (see, for instance (Daskalakis et al., 2020b) and (Daskalakis, 2022)), and do not contradict the non-existence of a local-VE since under this choice of parameters $\varepsilon = 0$ iff $\delta = 0$.

Remark 9.6.3 [Local convergence to ε-VE]**.**

As mentioned in Remark 9.4.5, the above finite-time global convergence result to ε-first-order VE, can be extended to a finite-time local convergence result to ε-first-order VE by instead applying Theorem 4.3.2 with the assumption that the initial iterate of the mirror extragradient learning dynamics starts close enough to a local weak solution of the VI $(\mathcal{X}^*, \boldsymbol{v})$. To the best of our knowledge this is the first finite-time local convergence result to ε-first-order VE in pseudo-games, as well as ε-first-order NE in games.

Remark 9.6.4 [Contributions to the literature]**.**

To the best of our knowledge, the above result is the first polynomial-time computation result for ε-first-order VE, as well as ε-first-order NE in variationally stable games. It is also the first and only existing non-asymptotic convergence analysis of the mirror extragradient learning dynamics in such games.

We note that for the choice of kernel function $h \doteq \|\cdot\|^2$ (i.e., the euclidean square norm), while Huang and Zhang (2023) do not explicitly prove the above result, the result could be inferred from Huang and Zhang's Theorem 3.16 when taken in conjuction with Lemma 9.6.1. As such, for this particular setting, our contribution can be seen as identifying Lemma 9.4.1 to apply Huang and Zhang's result to pseudo-games.

9.6.3 Merit Function Methods for First-Order GNE

Unfortunately, as the following example shows, beyond variationally stable pseudo-games it is in general not possible to guarantee the convergence of the mirror extragradient method to a first-order GNE.

Example 9.6.2 [Non-Convergence of First-Order Methods Beyond Variationally Stable Pseudo-Games]**.**
Consider the two player zero-sum game $(2, \mathcal{A}, \boldsymbol{u})$ where $\mathcal{A} \doteq \mathbb{R} \times \mathbb{R}$, and $u_1(a_1, a_2) = -u_2(a_1, a_2) = (a_2 - a_1)^2$. The set of first-order VE of this game are given by $\{(\boldsymbol{a}_1, \boldsymbol{a}_2) \in \mathbb{R} \times \mathbb{R} \mid \boldsymbol{a}_1 = \boldsymbol{a}_2\}$. Notice that for any $a_1^{(0)} > a_2^0$, for any choice of step sizes, the iterates generated by the mirror extragradient learning dynamics tend to negative infinity, while for $a_1^{(0)} < a_2^{(0)}$, the iterates tend to infinity.

To overcome this non-convergence issue, we will instead consider second-order methods. Our approach to derive a second order method method for pseudo-games will be to optimize a merit function associated with the first-order VE of any pseudo-game.[5] In particular, recall that by Lemma 9.6.1, the set of first-order VE of any pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ can be expressed as the set of strong solutions of the VI $(\mathcal{X}^*, \boldsymbol{v})$. As such, we will consider optimizing the regularized primal gap function associated with $(\mathcal{X}^*, \boldsymbol{v})$ which we call the variational exploitability.

Definition 9.6.1 [Variational Exploitability]**.**
Given a regularization parameter $\alpha \geq 0$, the α-variational exploitability $\Xi_\alpha : \mathcal{X}^* \to \mathbb{R}$ of any pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ is defined as:

$$\Xi_\alpha(\boldsymbol{a}) \doteq \max_{\boldsymbol{b} \in \mathcal{X}^*} \langle \boldsymbol{v}(\boldsymbol{a}), \boldsymbol{a} - \boldsymbol{b} \rangle - \frac{\alpha}{2} \|\boldsymbol{b} - \boldsymbol{a}\|^2 \tag{9.2}$$

The 0-variational exploitability is simply called the variational exploitability.

We note the following corollary of Lemma 4.4.1 for the variational exploitability which confirms that it is a merit function for first-order VE.

Corollary 9.6.2 [Properties of the regularized primal gap]**.**
Consider a continuous VI $(\mathcal{X}^*, \boldsymbol{v})$. Suppose that $\alpha > 0$, then $\max_{\boldsymbol{b} \in \mathcal{X}^*} \langle \boldsymbol{v}(\boldsymbol{a}), \boldsymbol{a} - \boldsymbol{b} \rangle - \frac{\alpha}{2} \|\boldsymbol{b} - \boldsymbol{a}\|^2$ has a unique solution. In addition, the following holds:

[5] The definition of merit functions for VE intuitively extends to first-order VE, by replacing any mentions of "VE" in Definition 9.4.6 with "first-order-VE".

1. $\boldsymbol{b}^*(\boldsymbol{a}) = \arg\max_{\boldsymbol{b}\in\mathcal{X}^*} \langle \boldsymbol{v}(\boldsymbol{a}), \boldsymbol{a} - \boldsymbol{b}\rangle - \frac{\alpha}{2}\left\|\boldsymbol{b} - \boldsymbol{a}\right\|^2 \doteq \Pi_{\mathcal{X}^*}\left[\boldsymbol{a} - \frac{1}{\alpha}\boldsymbol{v}(\boldsymbol{a})\right]$,

2. $\nabla\Xi_\alpha(\boldsymbol{a}) = \boldsymbol{v}(\boldsymbol{a}) - (\nabla\boldsymbol{v}(\boldsymbol{a}) + \alpha\mathbb{I})\left(\boldsymbol{b}^*(\boldsymbol{a}) - \boldsymbol{a}\right)$,

3. $\Xi_\alpha(\boldsymbol{a}) = \max_{\boldsymbol{b}\in\mathcal{X}^*} \frac{\alpha}{2}\left[\left\|\frac{1}{\alpha}\boldsymbol{v}(\boldsymbol{a})\right\|^2 - \left\|\boldsymbol{b} - \left(\boldsymbol{a} - \frac{1}{\alpha}\boldsymbol{v}(\boldsymbol{a})\right)\right\|^2\right]$

4. For all $\boldsymbol{a}\in\mathcal{X}^*$, $\Xi_\alpha(\boldsymbol{a}) \geq 0$ and $\Xi_\alpha(\boldsymbol{a}) = 0$ iff $\boldsymbol{a}$ is first-order VE.

Now, notice that we can minimize variational exploitability via a mirror descent method, but as the gradient $\nabla\Xi_\alpha$ involves $\boldsymbol{v}(\boldsymbol{a})$ and $\nabla\boldsymbol{v}(\boldsymbol{a})$ which both respectively depend on the gradient and hessian of the utility functions of the players, the arising method which we call the mirror variational learning dynamics (Algorithm 10) is a second-order learning dynamic.

Algorithm 10 Mirror Variational Learning Dynamics

Input: $\Xi_\alpha, h, \tau, \eta, \boldsymbol{a}^{(0)}$

Output: $\{\boldsymbol{a}^{(t)}\}_t$

1: **for** $t = 1, \dots, \tau$ **do**

2: $\quad \boldsymbol{a}^{(t+1)} \leftarrow \arg\min_{\boldsymbol{a}\in\mathcal{X}^*}\left\{\left\langle\nabla\Xi_\alpha(\boldsymbol{a}^{(t)}), \boldsymbol{a} - \boldsymbol{a}^{(t)}\right\rangle + \frac{1}{2\eta}\mathrm{div}_h(\boldsymbol{a}, \boldsymbol{a}^{(t)})\right\}$

return $\{\boldsymbol{a}^{(t)}\}_t$

Now, since the mirror variational learning dynamic algorithm is an instance of the mirror potential method (Algorithm 5) applied to the VI $(\mathcal{X}^*, \boldsymbol{v})$, we obtain the following theorem as an application of Theorem 4.4.1.

Theorem 9.6.2 [Mirror Variational Learning Dynamics Convergence]**.**
Let $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$ be a jointly convex, λ-Lipschitz-smooth pseudo-game with $\nabla^2 u_i$ in addition β-Lipschitz-continuous for al $i\in[n]$, h a 1-strongly-convex kernel function, $\alpha \geq 0$, $\eta \in \left(0, \frac{1}{2(2\beta\alpha\mathrm{diam}(\mathcal{X}^*)^2+1+2\lambda)}\right]$, and $\boldsymbol{a}^{(0)}\in\mathcal{X}^*$.

Consider the mirror variational learning dynamics (Algorithm 10) is run with the variational exploitability Ξ_α associated with $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u})$, the kernel function h, an arbitrary time horizon $\tau\in\mathbb{N}$, the step size η, the initial iterate $\boldsymbol{a}^{(0)}$, and which outputs $\{\boldsymbol{a}^{(t)}\}_t$. The following convergence bound to a stationary point of Ξ_α then holds:

$$\min_{k=0,1,\dots,\tau-1}\max_{\boldsymbol{a}\in\mathcal{X}^*}\langle\nabla\Xi_\alpha(\boldsymbol{a}^{(k)}), \boldsymbol{a}^{(k)} - \boldsymbol{a}\rangle \leq \frac{2\Xi_\alpha(\boldsymbol{a}^{(0)})}{\tau}$$

In addition, let $\boldsymbol{a}_{\text{best}}^{(\tau)} \in \arg\min_{\boldsymbol{a}^{(k)}:k=0,\ldots,\tau-1} \max_{\boldsymbol{a}\in\mathcal{X}^*} \langle \nabla \Xi_\alpha(\boldsymbol{a}^{(k)}), \boldsymbol{a}^{(k)} - \boldsymbol{a}\rangle$, then, for some choice of $\tau \in O(\frac{1}{\varepsilon})$, $\boldsymbol{a}_{\text{best}}^{(\tau)}$ is a ε-stationary point of Ξ_α.

With this theorem in order, we conclude on a remark on its interpretation before turning to the applications of our results.

Remark 9.6.5 [When are stationary points global solutions]**.**
A corollary of Remark 4.4.1, is that stationary points of the variational exploitability correspond to VE when the pseudo-game is monotone. As such, the above result implies a VE can be computed in polynomial-time via the mirror variational learning dynamics in monotone pseudo-games.

Chapter 10

Arrow-Debreu Economies

10.1 Background

An **Arrow-Debreu economy** $(n, m, \mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$, denoted $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$ when clear from context, comprises a finite set of $m \in \mathbb{N}_+$ divisible **commodities** and $n \in \mathbb{N}_+$ **consumers**. Each consumer $i \in [n]$ is characterized by a **set of consumptions** $\mathcal{X}_i \subseteq \mathbb{R}^m$, an **endowment** of commodities $\boldsymbol{e}_i = (e_{i1}, \ldots, e_{im}) \in \mathbb{R}^n$, and a **utility function** $u_i : \mathbb{R}^m \to \mathbb{R}$ which for any **consumption** $\boldsymbol{x}_i \in \mathcal{X}_i$ describes the utility $u_i(\boldsymbol{x}_i)$ consumer i derives.[1] We define any collection of per-consumer consumptions $\boldsymbol{x} \doteq (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n) \in \mathcal{X}$ a **consumption profile**, where $\mathcal{X} \doteq \bigtimes_{i \in [n]} \mathcal{X}_i$ is the **set of consumption profiles**, and any collection of per-consumer endowments an **endowment profile** $\boldsymbol{e} \doteq (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n) \in \mathbb{R}^{nm}$.

Remark 10.1.1.

For ease of exposition, without loss of generality, we restrict ourselves to Arrow-Debreu exchange economies and opt to not present Arrow-Debreu competitive economies (see Arrow and Debreu (1954)) which in addition to consumers also contain firms. Nevertheless, our focus on Arrow-Debreu exchange economies is without loss of generality since any firm can be represented as a consumer in an Arrow-Debreu exchange economy by adding an additional commodity into the economy which represents ownership of the firm, setting the consumption space of the new consumer to be equal to the production space of the firm, and its utility function so that it seeks to maximize its consumption of the commodity associated with the firm's ownership. The commodity associated with ownership of the firm should further appear in the

[1] In line with the literature (see, for instance, (Debreu et al., 1954)), the value of this utility function should not be interpreted to have any meaning, and the utility function u_i should be understood to represent a preference relation $\succeq_i$ on the space of consumptions $\mathcal{X}_i$ so that for any two consumptions $\boldsymbol{x}_i, \boldsymbol{x}'_i \in \mathcal{X}$, $u_i(\boldsymbol{x}_i) \geq u_i(\boldsymbol{x}'_i) \implies \boldsymbol{x}_i \succeq_i \boldsymbol{x}'_i$.

endowments of consumers that are supposed to have a contractual claim over the profits of the firms. A similar, albeit much more complicated reduction than described here was proposed earlier by Garg and Kannan (2015) which refer the reader to for additional details.

Any Arrow-Debreu economy $(n, m, \mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$ can be represented as a Walrasian economy $(m, \mathcal{Z})$ where the the excess demand correspondence $\mathcal{Z} : \Delta_m \rightrightarrows \mathbb{R}^m$ is given by:

$$\mathcal{Z}(\boldsymbol{p}) \doteq \sum_{i \in [n]} \left[\operatorname*{arg\,max}_{\boldsymbol{x}_i \in \mathcal{X}_i : \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p}} u_i(\boldsymbol{x}_i) \right] - \sum_{i \in [n]} \boldsymbol{e}_i \tag{10.1}$$

With this equivalence in hand, we can then obtain an equilibrium definition for Arrow-Debreu economies.

Definition 10.1.1.

Given $\varepsilon \geq 0$, a $\varepsilon-$**Arrow-Debreu equilibrium** $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is a tuple comprising consumptions $\boldsymbol{x}^* \in \mathbb{R}_+^{n \times n}$ and prices $\boldsymbol{p}^* \in \Delta_m$ s.t.

(Utility maximization) all consumers $i \in [n]$, $\varepsilon-$maximize their utility constrained by the value of their endowment: $\max_{\boldsymbol{x}_i \in \mathcal{X}_i : \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^*} u_i(\boldsymbol{x}_i) \leq u_i(\boldsymbol{x}_i^*) - \varepsilon$;

(Feasibility) the consumptions are $\varepsilon-$feasible, i.e., $\sum_{i \in [n]} \boldsymbol{x}_i^* \leq \sum_{i \in [n]} \boldsymbol{e}_i - \varepsilon$

(Walras' law) the value of the demand and the supply are equal, i.e., $\boldsymbol{p}^* \cdot \left(\sum_{i \in [n]} \boldsymbol{x}_i^* - \sum_{i \in [n]} \boldsymbol{e}_i\right) = 0$

A 0-Arrow-Debreu equilibrium is simply called an Arrow-Debreu equilibrium.

Remark 10.1.2 [Arrow-Debreu equilibrium prices are Walrasian equilibria]**.**

The set of Arrow-Debreu equilibrium prices of $(n, m, \mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$ is equal to the set of Walrasian equilibria of the Walrasian economy $(m, \mathcal{Z})$. To see this, notice that for any Arrow-Debreu equilibrium $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ of $(n, m, \mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$, we have $\sum_{i \in [n]} \boldsymbol{x}_i^* \in \mathcal{Z}(\boldsymbol{p}^*)$. Hence, since $\sum_{i \in [n]} \boldsymbol{x}_i^*$ is feasible and satisfies Walras' law under $\boldsymbol{p}^*$, $\boldsymbol{p}^*$ must be a Walrasian equilibrium of the Walrasian economy $(m, \mathcal{Z})$.

10.2 Solution Concepts and Existence

Using the observation made in Remark 10.1.2 we can obtain the existence of Arrow-Debreu equilibrium prices as a corollary of the existence of Walrasian equilibria, which in turn implies the existence of Arrow-

Debreu equilibrium consumptions since for any fixed price the Arrow-Debreu equilibrium consumptions can under very mild assumption be shown to exist.

Nevertheless, a more economically meaningful proof existence can be obtained by leveraging a fundamental relationship, first observed in the seminal work of Arrow and Debreu (1954), between pseudo-games and Arrow-Debreu economies.

Definition 10.2.1 [Arrow-Debreu Pseudo-Game]**.**
Given an Arrow-Debreu economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$, we define the associated $(n+1)$-player **Arrow-Debreu pseudo-game** $(n+1, 1, \mathcal{A}, \boldsymbol{h}, \boldsymbol{u}')$, denoted $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u}')$ when clear from context, in which the first n players are called "consumers", and the $(n+1)^{th}$ player is called the "auctioneer" and where:

(Action spaces) For all consumers $i \in [n]$, $\mathcal{A}_i \doteq \mathcal{X}'_i$ and for the auctioneer $\mathcal{A}_{n+1} \doteq \Delta_m$ where $\mathcal{X}'_i \doteq \left\{\boldsymbol{x}_i \mid \sum_{k\in[n]} \boldsymbol{x}_k \leq \sum_{k\in[n]} \boldsymbol{e}_k, \forall \boldsymbol{x} \in \mathcal{X}\right\}$ is the **restricted consumption space**

(Constraints) For all consumers $i \in [n]$, $h_i(\boldsymbol{x}, \boldsymbol{p}) = \boldsymbol{p} \cdot (\boldsymbol{e}_i - \boldsymbol{x}_i)$, and for the auctioneer $h_{n+1}(\boldsymbol{x}, \boldsymbol{p}) \doteq 0$

(Payoffs) For all consumers $i \in [n]$, $u'_i(\boldsymbol{x}, \boldsymbol{p}) \doteq u_i(\boldsymbol{x}_i)$, and for the auctioneer, $u'_{n+1}(\boldsymbol{x}, \boldsymbol{p}) \doteq \boldsymbol{p} \cdot \left(\sum_{i\in[n]} \boldsymbol{x}_i - \sum_{i\in[n]} \boldsymbol{e}_i\right)$

The Arrow-Debreu pseudo-game can succinctly be represented as the following $n+1$ simultaneous optimization problems:

$$\forall i \in [n], \qquad \max_{\boldsymbol{x}_i \in \mathcal{X}'_i : \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p}} u_i(\boldsymbol{x}_i) \qquad \Big| \qquad \max_{\boldsymbol{p} \in \Delta_m} \boldsymbol{p} \cdot \left(\sum_{i\in[n]} \boldsymbol{x}_i - \sum_{i\in[n]} \boldsymbol{e}_i\right)$$

Note that the above simultaneous $n+1$ optimization problems constitute a pseudo-game and not just a game since the prices chosen by the auctioneer determine the feasible action space of the consumers (i.e., **the budget set** of the consumers).

With the above pseudo-game in hand, we will show that the set of Arrow-Debreu equilibria of any Arrow-Debreu economy is equal to the set of GNE of the above pseudo-game. The results in this chapter will hold in the (quasi)concave Arrow-Debreu economies .

Definition 10.2.2 [(Quasi)concave economies]**.**

An Arrow-Debreu economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$ is said to be **quasiconcave (resp. concave)** iff it satisfies the following conditions for all consumers $i \in [n]$:

(Closed consumption set)	$\mathcal{X}_i$ is non-empty, bounded from below, closed, and convex
(Feasible budget set)	There exists a consumption that is strictly less than the consumer's endowment, i.e., for all $i \in [n]$, $\exists \boldsymbol{x}_i \in \mathcal{X}_i, \quad \boldsymbol{x}_i < \boldsymbol{e}_i$
(Continuity)	u_i is continuous
((Quasi)concavity)	u_i is quasiconcave (resp. concave)
(Non-satiation)	u_i is non-satiated, i.e., $\forall \boldsymbol{x}_i \in \mathcal{X}_i$, there exists $\boldsymbol{x}'_i \in \mathcal{X}_i$ s.t. $u_i(\boldsymbol{x}'_i) > u_i(\boldsymbol{x}'_i)$

Lemma 10.2.1 [GNE = Arrow-Debreu Equilibrium]**.**

The set of Arrow-Debreu equilibria of any quasiconcave Arrow-Debreu economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$ is equal to the set of GNE of the associated Arrow-Debreu pseudo-game.

Proof of Lemma 10.2.1

We prove only one direction (i.e., any GNE is an Arrow-Debreu equilibrium), the converse follows similarly. For additional details, see proof of Theorem 1 of Arrow and Debreu (1954).

Let $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ be any GNE of the Arrow-Debreu pseudo-game.

First, notice that by summing up the budget constraints of the consumers, by the definition of $\boldsymbol{p}^*$ at a GNE, we have:

$$
\begin{aligned}
0 &\geq \boldsymbol{p}^* \cdot \left(\sum_{i \in [n]} \boldsymbol{x}_i^* - \sum_{i \in [n]} \boldsymbol{e}_i \right) && \text{(Sum of consumer constraints)} \\
&= \max_{\boldsymbol{p} \in \Delta_m} \boldsymbol{p} \cdot \left(\sum_{i \in [n]} \boldsymbol{x}_i^* - \sum_{i \in [n]} \boldsymbol{e}_i \right) && \text{(Definition of } \boldsymbol{p}^*\text{)} \\
&\geq \boldsymbol{j}_j \cdot \left(\sum_{i \in [n]} \boldsymbol{x}_i^* - \sum_{i \in [n]} \boldsymbol{e}_i \right) && \forall j \in [m] \\
&= \sum_{i \in [n]} x_{ij}^* - \sum_{i \in [n]} e_{ij} && \forall j \in [m] \qquad (10.2)
\end{aligned}
$$

Hence, $\boldsymbol{x}^*$ is feasible

Second, suppose by contradiction that we had for some consumer $i \in [n]$:

$$\boldsymbol{x}_i^* \cdot \boldsymbol{p}^* < \boldsymbol{e}_i \cdot \boldsymbol{p}^* \tag{10.3}$$

Now, by non-satiation, there exists $\boldsymbol{x}_i' \in \mathcal{X}_i$ s.t. $u_i(\boldsymbol{x}_i') > u_i(\boldsymbol{x}_i^*)$. As a result, there must also exist $\lambda \in (0, 1)$ s.t. for the consumption $\boldsymbol{x}_i^\dagger \doteq \lambda \boldsymbol{x}_i' + (1 - \lambda)\boldsymbol{x}_i^*$, we have:

1. $\boldsymbol{x}_i^\dagger \in \mathcal{X}_i'$ since $\mathcal{X}_i'$ is convex and $\boldsymbol{x}_i^* \in \text{int}(\mathcal{X}_i')$ by Equation (10.3);
2. $u_i(\boldsymbol{x}_i^\dagger) > u_i(\boldsymbol{x}_i^*)$ since u_i is quasiconcave;
3. $\boldsymbol{x}_i^\dagger \cdot \boldsymbol{p}^* \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^*$ since the function $\boldsymbol{x}_i \mapsto \boldsymbol{x}_i \cdot \boldsymbol{p}^*$ is continuous.

However, this is a contradiction since $\boldsymbol{x}_i^* \in \underset{\boldsymbol{x}_i \in \mathcal{X}_i : \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p}}{\arg\max} u_i(\boldsymbol{x}_i)$.
Hence, for all consumers $i \in [n]$ we must have:

$$\boldsymbol{x}_i^* \cdot \boldsymbol{p}^* = \boldsymbol{e}_i^* \cdot \boldsymbol{p}^* \tag{10.4}$$

Summing the above across $i \in [n]$, and re-organizing the expression, we have:

$$\boldsymbol{p}^* \cdot \left(\sum_{i \in [n]} \boldsymbol{x}_i^* - \sum_{i \in [n]} \boldsymbol{e}_i \right) = 0$$

Hence, $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfies Walras' law.

Finally, since for all consumers $i \in [n]$, we have $\boldsymbol{x}_i^* \in \text{int}(\mathcal{X}_i')$ by Equation (10.2) and $\mathcal{X}_i' \subseteq \mathcal{X}_i$, this implies that $u_i(\boldsymbol{x}_i^*) = \max_{\boldsymbol{x}_i \in \mathcal{X}_i' : \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^*} u_i(\boldsymbol{x}_i) = \max_{\boldsymbol{x}_i \in \mathcal{X}_i : \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^*} u_i(\boldsymbol{x}_i)$. Hence, consumers are utility maximizing constrained by the value of their endowments at prices $\boldsymbol{p}^*$, i.e., for all consumers $i \in [n]$, we have $\boldsymbol{x}_i^* \in \underset{\boldsymbol{x}_i \in \mathcal{X}_i : \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^*}{\arg\max} u_i(\boldsymbol{x}_i)$.
Putting it all together, $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is then an Arrow-Debreu equilibrium.

Remark 10.2.1 [Bounded Excess Demand]**.**

Recall that in Remark 5.4.8 we had suggested that assuming boundedness of the excess demand for the convergence of mirror extratâtonnement (Algorithm 6) was a natural assumption. The above lemma provides a justification for this earlier remark since it means that in the definition of the excess demand for Arrow-Debreu markets we can replace $\mathcal{X}$ by $\mathcal{X}'$. Since unlike $\mathcal{X}$, $\mathcal{X}'$ is compact, this means that by Berge's maximum theorem (Berge, 1997) the excess demand $\boldsymbol{z}$ will be continuous over Δ_m,[2] and as such the excess demand is guaranteed to be bounded by $\max_{\boldsymbol{p} \in \Delta_m} \|\boldsymbol{z}(\boldsymbol{p})\|$ where the maximum is well-defined since Δ_m is non-empty, compact, and $\boldsymbol{z}$ is continuous.

With the above lemma in hand, we can then apply Theorem 9.2.1 to prove the existence of an Arrow-Debreu equilibrium.

Theorem 10.2.1 [Existence of Arrow-Debreu Equilibrium]**.**
An Arrow-Debreu equilibrium is guaranteed to exist in any quasiconcave Arrow-Debreu economy.

Proof of Theorem 10.2.1

Given any concave Arrow-Debreu economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$, construct the associated Arrow-Debreu pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u}')$ as in Definition 10.2.1. First, notice that the action spaces of the players $\{\mathcal{X}'_i\}_i$ and Δ_m are non-empty, compact, and convex. Second, $\boldsymbol{h}$ is continuous, for all players $i \in [n+1]$ $\boldsymbol{h}_i$ is quasiconcave in the action of the i^{th} player's action, and for all consumers $i \in [n]$, and $\boldsymbol{p} \in \Delta_m$ there exists $\boldsymbol{x}_i$ s.t. $\boldsymbol{h}_i(\boldsymbol{x}, \boldsymbol{p}) \geq 0$. Finally, for all players $i \in [n+1]$, u'_i is continuous, as well as quasiconcave in each player's action. Hence, by Theorem 9.2.1 a GNE of $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u}')$ is guaranteed to exist. In turn, by Remark 10.2.1 an Arrow-Debreu equilibrium is guaranteed to exist the Arrow-Debreu economy.

[2] Recall that for our algorithmic results we assume $\mathcal{Z}$ is singleton-valued.

10.3 Computation of Arrow-Debreu Equilibrium

With the question of existence of Arrow-Debreu equilibria out of the way, we now turn our attention to the computation of an Arrow-Debreu equilibrium.

To motivate some of the issues that arise in the computation of an Arrow-Debreu equilibrium, we will first apply the results derived in Chapter 5 establish the polynomial-time convergence of the mirror extratâtonnment process to an Arrow-Debreu equilibrium in Arrow-Debreu economies which satisfy the Minty condition assuming access to an excess demand oracle. However, as the implementation of an excess demand oracle is in general not possible, and the mirror extratâtonnment process is not guaranteed to converge with an approximate excess demand oracle, this polynomial-time computation cannot be interpreted

we will then introduce a new market dynamic we call the mirror extratrade dynamic which we will show converges in polynomial-time to an Arrow-Debreu equilibrium in a large class of Arrow-Debreu economies known as pure exchange economies.

10.3.1 Computational Model

With the question of existence answered, we now turn our attention to the computation of Arrow-Debreu equilibrium.

Algorithms for the computation of an Arrow-Debreu equilibrium are called **market dynamics**.

Definition 10.3.1 [Market Dynamics]**.**
Given an Arrow-Debreu economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$, and an initial iterate $(\boldsymbol{p}^{(0)}, \boldsymbol{x}^{(0)}) \in \Delta_m \times \mathcal{X}$, a **market dynamic $\boldsymbol{\pi}$** consists of an update function which generates the sequence of iterates $\{\boldsymbol{p}^{(t)}, \boldsymbol{x}^{(t)}\}_t$ given for all $t = 0, 1, \ldots$ by:

$$(\boldsymbol{p}^{(t+1)}, \boldsymbol{x}^{(t+1)}) \doteq \boldsymbol{\pi}\left((\mathcal{X}, \boldsymbol{e}, \boldsymbol{u}) \cup \bigcup_{i=0}^{t} (\boldsymbol{p}^{(i)}, \boldsymbol{x}^{(i)})\right)$$

The computational complexity results in this chapter will rely on the following computational model.

Definition 10.3.2 [Arrow-Debreu Economy Computational Model]**.**
Given an Arrow-Debreu economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$ s.t. for some $k \in \mathbb{N}_{++}$ the derivatives $\{\nabla^j \boldsymbol{u}\}_{j=1}^{k-1}$ are well-defined,

the computational complexity of a market dynamic is measured in term of the number of evaluations of the the functions $\boldsymbol{u}, \nabla \boldsymbol{u}, \ldots, \nabla^k \boldsymbol{u}$.

10.3.2 Tâtonnement in WARP Arrow-Debreu Economies

Recall that any Arrow-Debreu economy $(n, m, \mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$ can be represented as a Walrasian economy $(m, \mathcal{Z})$. This equivalence then provides us with a first approach to compute an Arrow-Debreu economy, namely computing a Walrasian equilibrium $\boldsymbol{p}^* \in \Delta_m$ of the Walrasian economy $(m, \mathcal{Z})$ using the mirror extratâtonnement process introduced in Chapter 5, and then setting for all consumers $i \in [n]$, $\boldsymbol{x}_i^* \dot{\in} \arg\max_{\boldsymbol{x}_i \in \mathcal{X}_i : \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^*} u_i(\boldsymbol{x}_i)$, so that $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is an Arrow-Debreu equilibrium of the Arrow-Debreu economy. Now, if we assume, that for all players $i \in [n]$, u_i is strictly concave, we can ensure that $\mathcal{Z} = \{\boldsymbol{z}\}$ is singleton-valued, and under suitable additional conditions we can further ensure that $\mathcal{Z}$ satisfies variational stability. However, recall that in Chapter 5, to prove the convergence of the mirror *extratâtonnement* process, we had to ensure that $\boldsymbol{z}$ is bounded, which in general is not true without highly restrictive assumptions on $\boldsymbol{z}$. Nevertheless, Lemma 10.2.1 suggests that we can consider an alternative excess demand correspondence definition $\mathcal{Z}'$ for Arrow-Debreu economies s.t. $\mathcal{Z}'$ is bounded-valued, which defines a Walrasian Arrow-Debreu economy we call **restricted Walrasian Arrow-Debreu economy**.

Definition 10.3.3 [Restricted Walrasian Arrow-Debreu Economy]**.**
Given an Arrow-Debreu economy $(n, m, \mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$, the **Walrasian Arrow-Debreu economy** $(m, \mathcal{Z})$ is a Walrasian economy with the excess demand correspondence given as:

$$\mathcal{Z}'(\boldsymbol{p}) = \sum_{i \in [n]} \left[\arg\max_{\boldsymbol{x}_i \in \mathcal{X}'_i : \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p}} u_i(\boldsymbol{x}_i) \right] - \sum_{i \in [n]} \boldsymbol{e}_i \ ,$$

where $\mathcal{X}'_i = \left\{ \boldsymbol{x}_i \mid \sum_{k \in [n]} \boldsymbol{x}_k \leq \sum_{k \in [n]} \boldsymbol{e}_k, \forall k \in [n], \boldsymbol{x}_k \in \mathcal{X}_k \right\}$.

Notice that in the definition of $\mathcal{Z}'$, the consumption sets $\{\mathcal{X}_i\}_i$ in the definition of $\mathcal{Z}$ (Equation (10.1)) are replaced by the restricted consumption sets $\{\mathcal{X}'_i\}_i$. From Lemma 10.2.1, we can then infer that any Walrasian equilibrium $\boldsymbol{p}^* \in \Delta_m$ of the restricted Walrasian Arrow-Debreu economy $(m, \mathcal{Z}')$ is an Arrow-Debreu equilibrium price of $(n, m, \mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$. Further, as shown in the following lemma, it is straightforward to verify that the Walrasian economy $(m, \mathcal{Z})$, as the name suggests, gives rise to a Walrasian competitive economy.

Lemma 10.3.1 [Arrow-Debreu Economies are Walrasian competitive Economies]**.**

Consider the Walrasian Arrow-Debreu competitive economy $(m, \mathcal{Z})$ associated with the quasiconcave Arrow-Debreu economy $(n, m, \mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$. Then, $\mathcal{Z}$ satisfies the following:

1. (Homogeneity of degree 0) For all $\lambda > 0$, $\mathcal{Z}(\lambda \boldsymbol{p}) = \mathcal{Z}(\boldsymbol{p})$
2. (Weak Walras' law) For all $\boldsymbol{p} \in \mathbb{R}^m_+$ and $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p})$, $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) \leq 0$
3. (Non-Satiation) for all $\boldsymbol{p} \in \mathbb{R}^m_+$, and $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p})$, $\boldsymbol{z}(\boldsymbol{p}) \leq \boldsymbol{0}_m \implies \boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) = 0$
4. (Continuity) The excess demand correspondence $\mathcal{Z}$ is upper hemicontinuous on Δ_m, non-empty-, compact-, and convex-valued.
5. (Boundedness) For all $\boldsymbol{p} \in \mathbb{R}^m_+$, and $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p})$, $\|\boldsymbol{z}(\boldsymbol{p})\|_\infty < \infty$

That is, the Walrasian Arrow-Debreu competitive economy $(m, \mathcal{Z})$ associated with the Arrow-Debreu economy $(n, m, \mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$, is a continuous competitive economy which is bounded.

Proof of Lemma 10.3.1

Homogeneity. For all $\lambda > 0$, we have:

$$\begin{aligned}\mathcal{Z}(\lambda \boldsymbol{p}) &= \sum_{i \in [n]} \left[\mathop{\arg\max}_{\boldsymbol{x}_i \in \mathcal{X}'_i : \boldsymbol{x}_i \cdot (\lambda \boldsymbol{p}) \leq \boldsymbol{e}_i \cdot (\lambda \boldsymbol{p})} u_i(\boldsymbol{x}_i) \right] - \sum_{i \in [n]} \boldsymbol{e}_i \\ &= \sum_{i \in [n]} \left[\mathop{\arg\max}_{\boldsymbol{x}_i \in \mathcal{X}'_i : \lambda \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \lambda \boldsymbol{e}_i \cdot \boldsymbol{p}} u_i(\boldsymbol{x}_i) \right] - \sum_{i \in [n]} \boldsymbol{e}_i \\ &= \sum_{i \in [n]} \left[\mathop{\arg\max}_{\boldsymbol{x}_i \in \mathcal{X}'_i : \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p}} u_i(\boldsymbol{x}_i) \right] - \sum_{i \in [n]} \boldsymbol{e}_i = \mathcal{Z}(\boldsymbol{p})\end{aligned}$$

Walras' law. Fix any $\boldsymbol{p} \in \mathbb{R}^m_+$, and let for all consumers $i \in [n]$, $\boldsymbol{x}^*_i \in \mathop{\arg\max}_{\boldsymbol{x}_i \in \mathcal{X}'_i : \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p}} u_i(\boldsymbol{x}_i)$. Then, we have:

$$\boldsymbol{x}^*_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p}$$

Summing up across all consumers, and re-organizing, we have:

$$\boldsymbol{p} \cdot \left(\sum_{i \in [n]} \boldsymbol{x}^*_i - \sum_{i \in [n]} \boldsymbol{e}_i \right) \leq 0$$

Hence, we have for all $\boldsymbol{p} \in \mathbb{R}^m_+$ and $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p})$, $\boldsymbol{p} \cdot \boldsymbol{z}(\boldsymbol{p}) \leq 0$.

Non-Satiation Fix any $\boldsymbol{p} \in \Delta_m$, and let for all consumers $i \in [n]$, $\boldsymbol{x}_i^* \in \arg\max_{\boldsymbol{x}_i \in \mathcal{X}'_i : \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p}} u_i(\boldsymbol{x}_i)$. Suppose by contradiction that $\boldsymbol{z}(\boldsymbol{p}) \leq \boldsymbol{0}_m$ but there exists some consumer $i \in [n]$ s.t.:

$$\boldsymbol{x}_i^* \cdot \boldsymbol{p} < \boldsymbol{e}_i \cdot \boldsymbol{p}$$

Now, by non-satiation, there exists $\boldsymbol{x}'_i \in \mathcal{X}_i$ s.t. $u_i(\boldsymbol{x}'_i) > u_i(\boldsymbol{x}_i^*)$. As a result, there must also exist $\lambda \in (0, 1)$ s.t. for the consumption $\boldsymbol{x}_i^\dagger \doteq \lambda \boldsymbol{x}'_i + (1 - \lambda)\boldsymbol{x}_i^*$, we have:

1. $\boldsymbol{x}_i^\dagger \in \mathcal{X}'_i$ since $\boldsymbol{x}_i^* \in \operatorname{int}(\mathcal{X}'_i)$;
2. $u_i(\boldsymbol{x}_i^\dagger) > u_i(\boldsymbol{x}_i^*)$ since u_i is quasiconcave;
3. $\boldsymbol{x}_i^\dagger \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p}$ since the function $\boldsymbol{x}_i \mapsto \boldsymbol{x}_i \cdot \boldsymbol{p}$ is continuous.

However, this is a contradiction since $\boldsymbol{x}_i^* \in \arg\max_{\boldsymbol{x}_i \in \mathcal{X}'_i : \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p}} u_i(\boldsymbol{x}_i)$.
Hence, for all consumers $i \in [n]$ we must have:

$$\boldsymbol{x}_i^* \cdot \boldsymbol{p} = \boldsymbol{e}_i^* \cdot \boldsymbol{p} \tag{10.5}$$

Summing the above across $i \in [n]$, and re-organizing the expression, we have for all $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p})$:

$$0 = \boldsymbol{p}^* \cdot \left(\sum_{i \in [n]} \boldsymbol{x}_i^* - \sum_{i \in [n]} \boldsymbol{e}_i \right) = \boldsymbol{p}^* \cdot \boldsymbol{z}(\boldsymbol{p})$$

Continuity.

Since $\mathcal{X}'$ is non-empty, compact, and convex, and for all consumers $i \in [n]$, u_i is continuous and quasiconcave, and $\exists \boldsymbol{x}_i \in \mathcal{X}_i, \quad \boldsymbol{x}_i < \boldsymbol{e}_i$ the assumptions of Berge's maximum theorem (Berge, 1997) hold, and the excess demand $\mathcal{Z}$ is upper hemicontinuous, non-empty, compact, and convex-valued over Δ_m.

Boundedness Since for all consumers $i \in [n]$, $\mathcal{X}_i$ is bounded from below, $\mathcal{X}'_i$ must be bounded as it is bounded from above by $\sum_{i \in [n]} \boldsymbol{e}_i$. Hence, for all consumers $i \in [n]$, $\mathcal{X}'_i$ is compact. Hence, we must have for all $\boldsymbol{p} \in \mathbb{R}^m_+$, and $\boldsymbol{z}(\boldsymbol{p}) \in \mathcal{Z}(\boldsymbol{p})$, $\|\boldsymbol{z}(\boldsymbol{p})\|_\infty < \operatorname{diam}(\mathcal{X}'_i)$.

With the above lemma in hand, we can then apply Theorem 5.4.1 to show convergence of the mirror extratâtonnement process under the suitable conditions derived in Chapter 5. Nevertheless, this result would rely on the existence of an exact excess demand oracle which in general cannot be guaranteed to exist. As such, to obtain a truly polynomial-time market dynamic for Arrow-Debreu economies we have to instead resort to solving not the restricted Walrasian Arrow-Debreu economy but rather the Arrow-Debreu pseudo-game.

10.3.3 Mirror Extratrade Dynamics in Pure Exchange Economies

Unfortunately, while the Arrow-Debreu pseudo-game allows us to establish the existence of an Arrow-Debreu equilibrium in Arrow-Debreu economies, as this pseudo-game does not have jointly convex constraints we cannot apply any of our theoretical results for showing convergence of learning dynamics in this pseudo-game. To get around this issue, we will restrict our attention to pure exchange economies, and introduce an alternative formulation of Arrow-Debreu economies as a jointly convex pseudo-game.

Definition 10.3.4 [Pure Exchange Economy]**.**
A **pure exchange economy** is an Arrow-Debreu economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$ where for all consumers $i \in [n]$:

(No debts payable) Endowments are positive, i.e., $\boldsymbol{e}_i \in \mathbb{R}^m_+$;

(No commodity creation) Consumers cannot create any commodity, i.e., $\mathcal{X}_i \subseteq \mathbb{R}^m_+$;

Intuitively, pure exchange economies are those Arrow-Debreu economies in which consumers do not owe any amount of any commodity to another consumer, and cannot create more commodities. We make the following observation about pure exchange economies.

Remark 10.3.1 [Positive supply of commodities]**.**
Note that in quasiconcave pure exchange economies, every commodity has a strictly positive endowment, i.e, for all commodities $j \in [m]$, $\sum_{k \in [n]} e_{kj} > 0$ and as such the above Hadamard divisions by $\sum_{k \in [n]} \boldsymbol{e}_k$ are all well-defined. To see this, recall that in Arrow-Debreu economies for all consumers $i \in [n]$, there exists $\boldsymbol{x}_i \in \mathcal{X}_i$ s.t. for all commodities $j \in [m]$, $x_{ij} < e_{ij}$. This last condition is however only guaranteed to hold in pure exchange economies iff for all consumers $i \in [n]$, $e_{ij} > 0$ since in pure exchange economies $\mathcal{X}_i \subseteq \mathbb{R}^m_+$, $e_{ij} \in \mathbb{R}^m_+$.

In light of the above remark, we note that we will make the following simplifying assumption without loss of generality which will lighten our notation going forward.

Assumption 10.3.1 [Normalized Aggregate Supply]**.**
Without loss of generality, for any pure exchange economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$ suppose that every commodity has unit aggregate supply, i.e., $\sum_{i\in[n]} \boldsymbol{e}_i = \mathbf{1}_m$.

The following remark explains why this assumption is without loss of generality.

Remark 10.3.2 [Unit Aggregate Supply]**.**
For convenience define $\oslash$ as the Hadamard division operator, i.e., $\boldsymbol{a} \oslash \boldsymbol{b} \doteq (a_i/b_i)_i$.

The assumption that every commodity has unit aggregate supply is without loss of generality since commodities are divisible and as such any pure exchange economy without aggregate unit supply $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$ can be converted into a pure exchange economy $(\mathcal{X}, \boldsymbol{e}', \boldsymbol{u}')$ with aggregate unit supply where for all consumers $i \in [n]$, $\boldsymbol{e}_i \doteq \boldsymbol{e}_i \oslash \sum_{k\in[n]} \boldsymbol{e}_k$ and $u'_i(\boldsymbol{x}_i) \doteq u_i\left(\boldsymbol{x}_i \oslash \sum_{k\in[n]} \boldsymbol{e}_k\right)$. Then, any Arrow-Debreu equilibrium $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ of $(\mathcal{X}, \boldsymbol{e}', \boldsymbol{u}')$ can be converted to an Arrow-Debreu equilibrium $(\boldsymbol{x}^{**}, \boldsymbol{p}^*)$ of $(\mathcal{X}, \boldsymbol{e}', \boldsymbol{u}')$ by simply setting for all consumers $i \in [n]$ and commodities $j \in [m]$, $x^{**}_{ij} \doteq x^*_{ij}(\sum_i e'_{ij})$. A straightforward algebraic manipulation then verifies this constructed Arrow-Debreu equilibrium satisfies feasibility, Walras' law, and utility maximization.

Intuitively, this construction tells us that Arrow-Debreu equilibrium consumptions of any pure exchange economy with unit aggregate supply can be interpreted as equilibrium equilibrium percentages of the aggregate supply consumed by the consumers in any pure exchange economy without aggregate unit supply.

We now introduce an alternative pseudo-game formulation of pure exchange economies which in contrast has jointly convex constraints. The pseudo-game we propose is equivalent to the Trading Post game (Shapley and Shubik, 1977) proposed by Shapley and Shubik, up to a variable substitution.

Definition 10.3.5 [Trading Post Pseudo-Game]**.**
For convenience define $\oslash$ as the Hadamard division operator, i.e., $\boldsymbol{a} \oslash \boldsymbol{b} \doteq (a_i/b_i)_i$.

Given a pure exchange economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$, we define the associated n-player **trading post pseudo-game** $(n, m+3, \mathcal{A}, \boldsymbol{h}, \boldsymbol{u}')$, denoted $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u}')$ when clear from context, in which the players are called consumers, and where:

(Action spaces) For all consumers $i \in [n]$, $\mathcal{A}_i \doteq \mathcal{B}_i = \left\{(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) \in \mathbb{R}^m \times \mathbb{R}^m_{++} \mid (\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i) \in \mathcal{X}'_i\right\}$ where $\mathcal{X}'_i$ is the restricted consumption set as defined in Definition 10.2.1.

(Constraints) For all consumers $i \in [n]$, $h_{i1}(\boldsymbol{\beta}, \boldsymbol{\pi}) = \boldsymbol{e}_i \cdot \left(\sum_{k\in[n]} \boldsymbol{\beta}_k\right) - \sum_{j\in[m]} \beta_{ij}$, $h_{i2}(\boldsymbol{\beta}, \boldsymbol{\pi}) = \sum_{i\in[n], j\in[m]} \beta_{ij} - 1$, $h_{i3}(\boldsymbol{\beta}, \boldsymbol{\pi}) = 1 - \sum_{i\in[n], j\in[m]} \beta_{ij}$ and $h_{i(j+1)}(\boldsymbol{\beta}, \boldsymbol{\pi}) \doteq \pi_{ij} - \sum_{k\in[n]} \beta_{kj}$ for all $j \in [m]$,

(Payoffs) For all consumers $i \in [n]$, $u'_i(\boldsymbol{\beta}, \boldsymbol{\pi}) \doteq u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i)$.

The trading post pseudo-game can succinctly be represented as the following n simultaneous optimization problems, s.t. for all $i \in [n]$:

$$\begin{aligned}
\max_{(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i)\in\mathcal{B}_i} \quad & u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i) & \\
\text{s.t.} \quad & \sum_{j\in[m]} \beta_{ij} = \boldsymbol{e}_i \cdot \left(\sum_{k\in[n]} \boldsymbol{\beta}_k\right) & \text{(Budget constraint)} \\
& \sum_{k\in[n]} \boldsymbol{\beta}_k \leq \boldsymbol{\pi}_i & \text{(Bid constraint)} \\
& \sum_{i\in[n], j\in[m]} \beta_{ij} = 1 & \text{(Bid constraint)}
\end{aligned}$$

With this definition in hand, we first show that set of Arrow-Debreu equilibria of any Arrow-Debreu equilibrium can be converted to GNE of the associated trading post game and vice-versa.

Lemma 10.3.2 [Trading Post GNE define Arrow-Debreu Equilibria]**.**
Consider a quasiconcave pure exchange economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$ and the associated trading post pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u}')$. Then, if $(\boldsymbol{\beta}^*, \boldsymbol{p}^*)$ is a GNE of $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u}')$ then $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is an Arrow-Debreu equilibrium of $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$ where we define:

$$\boldsymbol{p}^* \doteq \sum_{k\in[n]} \boldsymbol{\beta}^*_k \qquad\qquad \forall i \in [n], \qquad \boldsymbol{x}_i \doteq \boldsymbol{\beta}^*_i \oslash \boldsymbol{\pi}^*_i \ .$$

Proof of Lemma 10.3.2

Let $(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*)$ be a GNE of $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u}')$. Define $\boldsymbol{x}^*$ and $\boldsymbol{p}^*$ as follows:

$$\boldsymbol{p}^* \doteq \sum_{k \in [n]} \boldsymbol{\beta}_k^* \qquad \forall i \in [n], \quad \boldsymbol{x}_i \doteq \boldsymbol{\beta}_i^* \oslash \boldsymbol{\pi}_i^* \ .$$

First, note we have $\boldsymbol{x}_i = \boldsymbol{\beta}_i^* \oslash \boldsymbol{\pi}_i^* \in \mathcal{X}_i$, and $\boldsymbol{p}^* \in \Delta_m$ since $\sum_{j \in [m]} p_i^* = \sum_{k \in [n], j \in [m]} \beta_{kj} = 1$.

Second, summing $\boldsymbol{x}_i^*$ across all consumers $i \in [n]$ and using the bid constraint $(\sum_{k \in [n]} \boldsymbol{\beta}_k) \oslash (\sum_{k \in [n]} \boldsymbol{e}_k) \leq \boldsymbol{\pi}_i$, we have for all $j \in [m]$:

$$\sum_{i \in [n]} x_{ij}^* = \sum_{i \in [n]} \frac{\beta_{ij}^*}{\pi_{ij}^*} \leq \sum_{i \in [n]} \frac{\beta_{ij}^*}{\sum_{k \in [n]} \beta_{kj}} = \frac{\sum_{i \in [n]} \beta_{ij}^*}{\sum_{k \in [n]} \beta_{kj}} = \mathbf{1}_m = \sum_{i \in [n]} e_{ij} \tag{10.6}$$

Hence, $\boldsymbol{x}^* \leq \sum_{i \in [n]} \boldsymbol{e}_i$, and $\boldsymbol{x}^*$ is feasible.

Third, suppose by contradiction that fix some consumer $i \in [n]$ and commodity $j \in [m]$, we have $p_j^* > 0$, and $\sum_{k \in [n]} \beta_{kj}^* < \pi_{ij}^*$. Recall that, by non-satiation, there exists $\boldsymbol{x}_i' \in \mathcal{X}_i$ s.t. $u_i(\boldsymbol{x}_i') > u_i(\boldsymbol{x}_i^*)$. Let $\boldsymbol{\pi}_i' \in \mathbb{R}_{++}^m$, s.t. $\boldsymbol{x}_i' \doteq \boldsymbol{\beta}_i^* \oslash \boldsymbol{\pi}_i'$. Now, we can choose a small enough $\lambda \in (0, 1)$ s.t. for $\boldsymbol{x}_i^\dagger \doteq \mu \boldsymbol{x}_i' + (1 - \mu) \boldsymbol{x}_i^*$, we have:

1. $\boldsymbol{x}_i^\dagger \doteq \boldsymbol{\beta}_i^* \oslash \boldsymbol{\pi}_i^\dagger$ for some $\boldsymbol{\pi}_i^\dagger$ s.t. $(\boldsymbol{\beta}_i^*, \boldsymbol{\pi}_i^\dagger) \in \mathcal{B}_i$ since $(\boldsymbol{\beta}_i^*, \boldsymbol{\pi}_i^*) \in \operatorname{int}(\mathcal{B}_i)$ by Equation (10.6);
2. $\sum_{k \in [n]} \beta_{kj}^* \leq \pi_{ij}^\dagger$ since both sides of the inequality are continuous in $\boldsymbol{\pi}$;

Further, observe that by quasiconcavity, we have that $u_i(\boldsymbol{\beta}_i^* \oslash \boldsymbol{\pi}_i^\dagger) = u_i(\boldsymbol{x}_i^\dagger) > u_i(\boldsymbol{x}_i^*) = u_i(\boldsymbol{\beta}_i^* \oslash \boldsymbol{\pi}_i^*)$ since u_i is quasiconcave. However, this is in contradiction, to the definition of a GNE, hence we must have for any consumer $i \in [n]$ and commodity $j \in [m]$ s.t. $p_j^* > 0$, we have $\sum_{k \in [n]} \beta_{kj} = \pi_{ij}$.

Hence, for all consumers $i \in [n]$, we can have:

$$\sum_{j \in [m]} \beta_{ij} \leq \boldsymbol{e}_i \cdot \left(\sum_{k \in [n]} \boldsymbol{\beta}_k \right)$$

$$\sum_{j \in [m]} x_{ij}^* \pi_{ij}^* = \|\boldsymbol{\beta}^{(*)}\|_1 (\boldsymbol{e}_i \cdot \boldsymbol{p}^*)$$

$$\sum_{j \in [m]: p_j^* > 0} x_{ij}^* \pi_{ij}^* + \sum_{j \in [m]: p_j^* = 0} \underbrace{x_{ij}^*}_{=0} \pi_{ij}^* = \|\boldsymbol{\beta}^{(*)}\|_1 \boldsymbol{e}_i \cdot \boldsymbol{p}^*$$

$$\|\boldsymbol{\beta}^{(*)}\|_1 \sum_{j \in [m]: p_j^* > 0} x_{ij}^* p_j^* + \|\boldsymbol{\beta}^{(*)}\|_1 \sum_{j \in [m]: p_j^* = 0} x_{ij}^* p_j^* = \|\boldsymbol{\beta}^{(*)}\|_1 \boldsymbol{e}_i \cdot \boldsymbol{p}^*$$

$$\boldsymbol{x}_i^* \cdot \boldsymbol{p}^* = \boldsymbol{e}_i \cdot \boldsymbol{p}^*$$

Summing the above for all consumers $i \in [n]$, and re-reorganizing expressions, we have:

$$\boldsymbol{p}^* \cdot \left(\sum_{i \in [n]} \boldsymbol{x}_i^* - \sum_{i \in [n]} \boldsymbol{e}_i \right) = 0$$

Hence, $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ satisfies Walras' law.

Finally, since for all consumers $i \in [n]$, we have $\boldsymbol{x}_i^* \in \operatorname{int}(\mathcal{X}_i')$ by Equation (10.2) and $\mathcal{X}_i' \subseteq \mathcal{X}_i$, this implies that $u_i(\boldsymbol{x}_i^*) = \max_{\boldsymbol{x}_i \in \mathcal{X}_i' : \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^*} u_i(\boldsymbol{x}_i) = \max_{\boldsymbol{x}_i \in \mathcal{X}_i : \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^*} u_i(\boldsymbol{x}_i)$. Hence, consumers are utility maximizing constrained by the value of their endowments at prices $\boldsymbol{p}^*$, i.e., for all consumers $i \in [n]$, we have $\boldsymbol{x}_i^* \in \operatorname*{arg\,max}_{\boldsymbol{x}_i \in \mathcal{X}_i : \boldsymbol{x}_i \cdot \boldsymbol{p}^* \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^*} u_i(\boldsymbol{x}_i)$.

Putting it all together, $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ is then an Arrow-Debreu equilibrium.

Remark 10.3.3 [Arrow-Debreu equilibria define trading post GNE]**.**

While we do not present a statement and proof for the converse variant of the above lemma as we will not make use of it, using a similar argument to that provided in Lemma 10.3.2, we can also show that any Arrow-Debreu equilibrium can used to construct a trading post GNE. The converse direction is less useful at present since unlike in Lemma 10.2.1 where we have shown that the set of GNE of the Arrow-Debreu pseudo-game is equal to the set of Arrow-Debreu equilibria of the Arrow-Debreu economies, for the trading post pseudo-game we instead have an equivalence after suitable algebraic manipulation of the GNE of the trading post pseudo-game, and the Arrow-Debreu equilibrium of a quasiconcave pure exchange economy. That is, when taken in conjuction with its converse, the above result cannot be interpreted an "if and only if result but perhaps rather as a polynomial-time equivalence result.

With the above lemma in hand, we can now describe the properties of the trading post pseudo-game with the following result.

Lemma 10.3.3 [Trading Post Pseudo-Game Properties]**.**

The trading post pseudo-game associated with any quasiconcave pure exchange economy is a quasiconcave pseudo-game with jointly convex constraints.

Proof of Lemma 10.3.3

The trading post pseudo-game is jointly convex: First, note that for all consumers $i \in [n]$, $\mathcal{B}_i$ is non-empty, compact, convex since $\mathcal{X}'_i$ is non-empty, compact, and convex, and $\mathcal{B}_i$ is the perspective transformation of $\mathcal{X}'_i$ (see, for instance, Section 2.3.3 of Boyd et al. (2004)) which is continuous and preserves convexity.

Second, notice that the constraints of the pseudo-game are all affine in $(\boldsymbol{\beta}, \boldsymbol{\pi})$, and hence are continuous, and concave in $(\boldsymbol{\beta}, \boldsymbol{\pi})$. As such, the trading pseudo-game has jointly convex constraints.

The trading post pseudo-game is quasiconcave: Recall that in the trading pseudo-game, for all consumers $i \in [n]$, $u'_i(\boldsymbol{\beta}, \boldsymbol{\pi}) \doteq u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i)$. Now, u_i is quasiconcave iff for all $\alpha \in \mathbb{R}$ the superlevel sets $\mathcal{X}_i^\alpha \doteq \{\boldsymbol{x}_i \in \mathcal{X}_i \mid u_i(\boldsymbol{x}_i) \geq \alpha\}$ are convex. Now, if we pass the superlevel sets through the mapping $\mathcal{X}_i^\alpha \mapsto \{(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) \in \mathbb{R}^m_+ \times \mathbb{R}^m_{++} \mid \boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i \in \mathcal{X}_i^\alpha\}$, since the mapping is a linear-fractional transformation of $\mathcal{X}_i^\alpha$, for all $\alpha \in \mathbb{R}$, the transformed sets are convex as well (see, for instance, Section 2.3.3 of Boyd et al. (2004)). That is, for all $\alpha \in \mathbb{R}$, the superlevel sets $\{(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) \in \mathbb{R}^m_+ \times \mathbb{R}^m_{++} \mid u'_i(\boldsymbol{\beta}, \boldsymbol{\pi}) \geq \alpha\} = \{(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) \in \mathbb{R}^m_+ \times \mathbb{R}^m_{++} \mid u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i) \geq \alpha\}$ of u'_i are convex. Hence, u'_i is quasiconcave.

Using the above lemma, we can in turn obtain the existence of a VE in the trading post pseudo-game as a corollary of Lemma 10.3.3 as it means that the trading post pseudo-game satisfies the conditions for the existence of a VE (Theorem 9.2.2).

Corollary 10.3.1.
The set of VE of the trading post pseudo-game associated with any Arrow-Debreu economy is non-empty.

Since any VE is a GNE, by Lemma 10.3.2 this means that we can set our sight on the computation of a VE of the trading post pseudo-game to compute an Arrow-Debreu equilibrium in pure exchange economies. Our first approach to computing a VE will be to apply the mirror extragradient learning dynamics to the trading post pseudo-game. We call the market dynamics arising from running the mirror extragradient learning dynamics on the trading post pseudo-game the **mirror extratrade dynamics**. Unfortunately, to guarantee convergence of the mirror extragradient learning dynamics in the trading post pseudo-game, we have to ensure that the trading post pseudo-game is variationally stable. Thankfully, we can show that when the utility functions of the consumers in a pure exchange economy are concave (rather than quasiconcave), the associated trading post pseudo-game is guaranteed to be variationally stable. To this end, we introduce the following technical lemma.

Lemma 10.3.4 [Pseudoconcavity of composition of concave and ratio functions]**.**
Let $f : \mathbb{R}^n_+ \to \mathbb{R}$ be a concave function, and $\boldsymbol{g} : \mathbb{R}^n_+ \times \mathbb{R}^n_{++} \to \mathbb{R}^n_+$ be the ratio function s.t. $\boldsymbol{g}(\boldsymbol{a}, \boldsymbol{b}) \doteq \boldsymbol{a} \oslash \boldsymbol{b}$. Then, $\nu(\boldsymbol{a}, \boldsymbol{b}) \doteq f(\boldsymbol{g}(\boldsymbol{a}, \boldsymbol{b}))$ is pseudoconcave.

Proof of Lemma 10.3.4

Suppose that $\boldsymbol{a} \in \mathbb{R}^n_+$ and $\boldsymbol{b} \in \mathbb{R}^n_{++}$ s.t.

$$0 \geq \max_{\boldsymbol{a}' \in \mathbb{R}^n_+, \boldsymbol{b}' \in \mathbb{R}^n_{++}} \left\langle \nabla \nu(\boldsymbol{a}, \boldsymbol{b}), (\boldsymbol{a}', \boldsymbol{b}') - (\boldsymbol{a}, \boldsymbol{b}) \right\rangle \quad .$$

Plugging in the above, $\nabla_{a_i} \nu(\boldsymbol{a}, \boldsymbol{b}) = \frac{1}{b_i} \nabla_i f\left(\boldsymbol{a} \oslash \boldsymbol{b}\right)$, and $\nabla_{b_i} \nu(\boldsymbol{a}, \boldsymbol{b}) = -\frac{a_i}{b_i^2} \nabla_i f\left(\boldsymbol{a} \oslash \boldsymbol{b}\right)$. Then, we have:

$$\begin{aligned}
0 &\geq \max_{\boldsymbol{a}' \in \mathbb{R}^n_+, \boldsymbol{b}' \in \mathbb{R}^n_{++}} \left\langle \nabla \nu(\boldsymbol{a}, \boldsymbol{b}), (\boldsymbol{a}', \boldsymbol{b}') - (\boldsymbol{a}, \boldsymbol{b}) \right\rangle \\
&= \max_{\boldsymbol{a}' \in \mathbb{R}^n_+, \boldsymbol{b}' \in \mathbb{R}^n_{++}} \sum_{i \in [n]} \left\langle \nabla_{(a_i, b_i)} \nu(\boldsymbol{a}, \boldsymbol{b}), (a_i', b_i') - (a_i, b_i) \right\rangle \\
&= \max_{\boldsymbol{a}' \in \mathbb{R}^n_+, \boldsymbol{b}' \in \mathbb{R}^n_{++}} \sum_{i \in [n]} \left\langle \left(\frac{1}{b_i} \nabla_i f\left(\boldsymbol{a} \oslash \boldsymbol{b}\right), -\frac{a_i}{b_i^2} \nabla_i f\left(\boldsymbol{a} \oslash \boldsymbol{b}\right) \right), (a_i', b_i') - (a_i, b_i) \right\rangle \\
&= \max_{\boldsymbol{a}' \in \mathbb{R}^n_+, \boldsymbol{b}' \in \mathbb{R}^n_{++}} \sum_{i \in [n]} \left[\frac{1}{b} \left\langle \nabla_i f\left(\boldsymbol{a} \oslash \boldsymbol{b}\right), a_i' - a_i \right\rangle - \frac{a_i}{b_i^2} \left\langle \nabla_i f\left(\boldsymbol{a} \oslash \boldsymbol{b}\right), b_i' - b_i \right\rangle \right] \\
&= \max_{\boldsymbol{a}' \in \mathbb{R}^n_+, \boldsymbol{b}' \in \mathbb{R}^n_{++}} \sum_{i \in [n]} \left[\left\langle \nabla_i f\left(\boldsymbol{a} \oslash \boldsymbol{b}\right), \frac{a_i'}{b_i} - \frac{a_i}{b_i} \right\rangle + \left\langle \nabla_i f\left(\boldsymbol{a} \oslash \boldsymbol{b}\right), \frac{a_i}{b_i} - \frac{a_i b_i'}{b_i^2} \right\rangle \right] \\
&= \max_{\boldsymbol{a}' \in \mathbb{R}^n_+, \boldsymbol{b}' \in \mathbb{R}^n_{++}} \sum_{i \in [n]} \left\langle \nabla_i f\left(\boldsymbol{a} \oslash \boldsymbol{b}\right), \frac{a_i'}{b_i} - \frac{a_i b_i'}{b_i^2} \right\rangle \\
&= \max_{\boldsymbol{a}' \in \mathbb{R}^n_+, \boldsymbol{b}' \in \mathbb{R}^n_{++}} \sum_{i \in [n]} \left\langle \nabla_i f\left(\boldsymbol{a} \oslash \boldsymbol{b}\right), \frac{b_i'}{b_i} \left(\frac{a_i'}{b_i'} - \frac{a_i}{b_i} \right) \right\rangle \\
&= \sum_{i \in [n]} \underbrace{\max_{a_i', b_i'} \left\{ \frac{b_i'}{b_i} \left\langle \nabla_i f\left(\boldsymbol{a} \oslash \boldsymbol{b}\right), \frac{a_i'}{b_i'} - \frac{a_i}{b_i} \right\rangle \right\}}_{\geq \frac{b_i}{b_i} \left\langle \nabla_i f(\boldsymbol{a} \oslash \boldsymbol{b}), \frac{a_i}{b_i} - \frac{a_i}{b_i} \right\rangle \geq 0}
\end{aligned}$$

Now, since for all $i \in [n]$, $\frac{b'_i}{b_i} > 0$, and $\max_{a'_i, b'_i} \left\{ \frac{b'_i}{b_i} \left\langle \nabla_i f\left(\boldsymbol{a} \oslash \boldsymbol{b}\right), \frac{a'_i}{b'_i} - \frac{a_i}{b_i} \right\rangle \right\} \geq 0$, for the above inequality to hold at $(\boldsymbol{a}, \boldsymbol{b})$, it must be that for all $\boldsymbol{a}' \in \mathbb{R}^n_+, \boldsymbol{b}' \in \mathbb{R}^n_{++}$:

$$\forall i \in [n], 0 \geq \left\langle \nabla_i f\left(\boldsymbol{a} \oslash \boldsymbol{b}\right), \frac{a'_i}{b'_i} - \frac{a_i}{b_i} \right\rangle \implies 0 \geq \sum_{i \in [n]} \left\langle \nabla_i f\left(\boldsymbol{a} \oslash \boldsymbol{b}\right), \frac{a'_i}{b'_i} - \frac{a_i}{b_i} \right\rangle$$

Hence, since f is concave, we must have for all :

$$\begin{aligned} 0 &\geq \sum_{i \in [n]} \left\langle \nabla_i f\left(\boldsymbol{a} \oslash \boldsymbol{b}\right), \frac{a'_i}{b'_i} - \frac{a_i}{b_i} \right\rangle \\ &= \left\langle \nabla f\left(\boldsymbol{a} \oslash \boldsymbol{b}\right), (\boldsymbol{a}' \oslash \boldsymbol{b}') - (\boldsymbol{a} \oslash \boldsymbol{b}) \right\rangle \\ &\geq f\left(\boldsymbol{a}' \oslash \boldsymbol{b}'\right)\right) - f\left(\boldsymbol{a} \oslash \boldsymbol{b}\right) \\ &= \nu(\boldsymbol{a}', \boldsymbol{b}') - \nu(\boldsymbol{a}, \boldsymbol{b}) \end{aligned}$$

Putting it all together, for all $\boldsymbol{a}, \boldsymbol{a}' \in \mathbb{R}^n_+$ and $\boldsymbol{b}, \boldsymbol{b}' \in \mathbb{R}^n_{++}$:

$$\left\langle \nabla \nu(\boldsymbol{a}, \boldsymbol{b}), (\boldsymbol{a}', \boldsymbol{b}') - (\boldsymbol{a}, \boldsymbol{b}) \right\rangle \implies \nu(\boldsymbol{a}, \boldsymbol{b}) \geq \nu(\boldsymbol{a}', \boldsymbol{b}')$$

That is, ν is pseudoconcave.

Lemma 10.3.5 [Variational Stability in the Trading Post Pseudo-Game]**.**
Consider a concave pure exchange economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$. Then, the trading post pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u}')$ associated with $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$ is pseudomonotone, and variationally stable.

Proof of Lemma 10.3.5

Let $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u}')$ be the the trading post pseudo-game associated with the pure exchange economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$ where for all consumers $i \in [n]$, u_i is concave.

First, for all consumers $i \in [n]$, u'_i depends only on $(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i)$, hence, we re-define $u'_i(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) \doteq u'_i(\boldsymbol{\beta}, \boldsymbol{\pi})$. Consider the utilitarian welfare function $w(\boldsymbol{x}) \doteq \sum_{i \in [n]} u_i(\boldsymbol{x}_i)$. Since for all consumers $i \in [n]$, u_i is concave, then w must also be concave as it is the sum of concave functions (see, for instance, Section 3.2 of Boyd et al. (2004)). If we then define $w'(\boldsymbol{\beta}, \boldsymbol{\pi}) \doteq w(\boldsymbol{\beta} \oslash \boldsymbol{\pi}) = \sum_{i \in [n]} u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i) = \sum_{i \in [n]} u'_i(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i)$. Now, by Lemma 10.3.4, we then must have that w' is pseudoconcave since it is the composition of a concave function and the ratio function. Since w' is pseudcoconcave, then $-w'$ is pseudoconvex, and hence $-\nabla w'$ must be pseudomonotone (see, for instance, Theorem 4.1 of (Aussel et al., 1994)). Further, notice that we have:

$$\nabla w'(\boldsymbol{\beta}, \boldsymbol{\pi}) = (\nabla u'_1(\boldsymbol{\beta}_1, \boldsymbol{\pi}_1), \ldots, \nabla u'_n(\boldsymbol{\beta}_n, \boldsymbol{\pi}_n))$$

To prove that the trading post pseudo-game is variationally stable (Definition 9.4.2), we have to show that there exists $(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*) \in \mathcal{X}^*$ s.t.

$$\sum_{i \in [n]} \left\langle \nabla u'_i(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i), (\boldsymbol{\beta}_i^*, \boldsymbol{\pi}_i^*) - (\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) \right\rangle \geq 0 \qquad \forall (\boldsymbol{\beta}, \boldsymbol{\pi}) \in \mathcal{X}^*$$

Equivalently, define the pseudo-game operator associated with the trading post pseudo-game:

$$\boldsymbol{v}(\boldsymbol{\beta}, \boldsymbol{\pi}) \doteq -(\nabla u'_1(\boldsymbol{\beta}_1, \boldsymbol{\pi}_1), \ldots, \nabla u'_n(\boldsymbol{\beta}_n, \boldsymbol{\pi}_n))$$

Then, the trading post pseudo-game is variationally stable iff the set of weak solutions of the VI $(\mathcal{X}^*, \boldsymbol{v})$ is non-empty.

Notice that, $\boldsymbol{v} = -\nabla w'(\boldsymbol{\beta}, \boldsymbol{\pi})$. This implies that $(\mathcal{X}^*, \boldsymbol{v}) = (\mathcal{X}^*, -\nabla w')$. Since $-\nabla w'$ is pseudomonotone, and $\mathcal{X}^*$ is non-empty, and compact (Lemma 10.3.3), then a weak solution of $(\mathcal{X}^*, -\nabla w')$ is guaranteed to exist. As a result, a weak solution of the VI $(\mathcal{X}^*, \boldsymbol{v})$ is also guaranteed to exist. That is, there exists $(\boldsymbol{\beta}^*, \boldsymbol{\pi}^*) \in \mathcal{X}^*$ s.t.

$$\sum_{i \in [n]} \left\langle \nabla u'_i(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i), (\boldsymbol{\beta}_i^*, \boldsymbol{\pi}_i^*) - (\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) \right\rangle \geq 0 \qquad \forall (\boldsymbol{\beta}, \boldsymbol{\pi}) \in \mathcal{X}^* \ ,$$

With the sufficient conditions for the trading post pseudo-game to be variationally stable in hand, to guaranteed polynomial-time convergence of the mirror extragradient learning dynamics, we have to ensure that the trading post pseudo-game is Lipschitz-smooth. The following lemma provides necessary conditions to ensure the Lipschitz-smoothness of the trading post pseudo-game.

Lemma 10.3.6 [Lipschitz-smoothness of Trading Post Pseudo-Game]**.**
Consider a pure exchange economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$ s.t. for all consumers $i \in [n]$, u_i is ℓ-Lipschitz-continuous, and ∇u_i is λ-Lipschitz-continuous. Consider the trading post pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u}')$ associated with the pure

exchange economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$. Further, define:

$$\pi_{\min} \doteq \min\{\pi_{ij} : (\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) \in \mathcal{B}_i, \forall i \in [n], j \in [m]\} \enspace ,$$

$$\pi_{\max} \doteq \max\{\pi_{ij} : (\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) \in \mathcal{B}_i, \forall i \in [n], j \in [m]\} \enspace ,$$

$$\beta_{\max} \doteq \max\{\beta_{ij} : (\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) \in \mathcal{B}_i, \forall i \in [n], j \in [m]\} \enspace .$$

Then, for all consumers $i \in [n]$, u'_i is $m\left[\max\left(\frac{\lambda}{\pi_{\min}^2}, \frac{\beta'_{\max}\lambda}{\pi_{\min}^2}, \frac{\ell}{\pi_{\min}^2}\right) + \max\left(\frac{\pi_{\max}\lambda}{\pi_{\min}}, \frac{\pi_{\max}\lambda}{\pi_{\min}^2}, \ell\right)\right]$ $-$Lipschitz-smooth.

Proof of Lemma 10.3.6

First, note that we have for all consumers $i \in [n]$ and commodities $j \in [m]$:

$$\begin{aligned}\nabla_{\beta_{ij}} u'_i(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) &= \nabla_{\beta_{ij}} \left[u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i)\right] \\ &= \frac{\nabla_{x_{ij}} u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i)}{\pi_{ij}} \enspace ,\end{aligned}$$

and

$$\begin{aligned}\nabla_{\pi_{ij}} u'_i(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) &= \nabla_{\pi_{ij}} \left[u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i)\right] \\ &= -\pi_{ij}^2 \nabla_{x_{ij}} u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i) \enspace .\end{aligned}$$

Now, we have for all $i \in [n]$, $j \in [m]$ $(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i), (\boldsymbol{\beta}'_i, \boldsymbol{\pi}'_i) \in \mathcal{B}_i$:

$$\begin{aligned}&\left|\frac{\nabla_{x_{ij}} u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i)}{\pi_{ij}} - \frac{\nabla_{x_{ij}} u_i(\boldsymbol{\beta}'_i \oslash \boldsymbol{\pi}'_i)}{\pi'_{ij}}\right| \\ &= \left|\frac{1}{\pi_{ij}}\left(\nabla_{x_{ij}} u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i) - \nabla_{x_{ij}} u_i(\boldsymbol{\beta}'_i \oslash \boldsymbol{\pi}'_i)\right) + \nabla_{x_{ij}} u_i(\boldsymbol{\beta}'_i \oslash \boldsymbol{\pi}'_i)\left(\frac{1}{\pi_{ij}} - \frac{1}{\pi'_{ij}}\right)\right| \\ &\leq \left|\frac{1}{\pi_{ij}}\left(\nabla_{x_{ij}} u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i) - \nabla_{x_{ij}} u_i(\boldsymbol{\beta}'_i \oslash \boldsymbol{\pi}'_i)\right)\right| + \left|\nabla_{x_{ij}} u_i(\boldsymbol{\beta}'_i \oslash \boldsymbol{\pi}'_i)\left(\frac{1}{\pi_{ij}} - \frac{1}{\pi'_{ij}}\right)\right|\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{\pi_{ij}} \left| \left(\nabla_{x_{ij}} u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i) - \nabla_{x_{ij}} u_i(\boldsymbol{\beta}'_i \oslash \boldsymbol{\pi}'_i) \right) \right| + \left| \nabla_{x_{ij}} u_i(\boldsymbol{\beta}'_i \oslash \boldsymbol{\pi}'_i) \right| \left| \frac{1}{\pi_{ij}} - \frac{1}{\pi'_{ij}} \right| \\
&\leq \frac{\lambda}{\pi_{ij}} \left\| (\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i) - (\boldsymbol{\beta}'_i \oslash \boldsymbol{\pi}'_i) \right\| + \ell \left| \frac{1}{\pi_{ij}} - \frac{1}{\pi'_{ij}} \right| \\
&= \frac{\lambda}{\pi_{ij}} \left\| (\boldsymbol{\beta}_i - \boldsymbol{\beta}'_i) \oslash \boldsymbol{\pi}_i - \boldsymbol{\beta}'_i \odot \left(\mathbf{1}_m \oslash \boldsymbol{\pi}_i - \mathbf{1}_m \oslash \boldsymbol{\pi}'_i \right) \right\| + \ell \left| \frac{1}{\pi_{ij}} - \frac{1}{\pi'_{ij}} \right| \\
&\leq \frac{\lambda}{\pi_{ij}} \left\| (\boldsymbol{\beta}_i - \boldsymbol{\beta}'_i) \oslash \boldsymbol{\pi}_i \right\| + \left\| \boldsymbol{\beta}'_i \odot \left(\mathbf{1}_m \oslash \boldsymbol{\pi}_i - \mathbf{1}_m \oslash \boldsymbol{\pi}'_i \right) \right\| + \ell \left| \frac{1}{\pi_{ij}} - \frac{1}{\pi'_{ij}} \right| \\
&\leq \frac{\lambda}{\pi_{ij} \min_{k \in [m]} \{\pi_{ik}\}} \left\| \boldsymbol{\beta}_i - \boldsymbol{\beta}'_i \right\| + \lambda \left\| \boldsymbol{\beta}'_i \right\| \left\| \mathbf{1}_m \oslash \boldsymbol{\pi}_i - \mathbf{1}_m \oslash \boldsymbol{\pi}'_i \right\| + \ell \left| \frac{1}{\pi_{ij}} - \frac{1}{\pi'_{ij}} \right|
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\lambda}{\pi_{ij} \min_{k \in [m]} \{\pi_{ik}\}} \left\| \boldsymbol{\beta}_i - \boldsymbol{\beta}'_i \right\| + \lambda \left\| \boldsymbol{\beta}'_i \right\| \left\| \left(\boldsymbol{\pi}'_i - \boldsymbol{\pi}_i \right) \oslash \left(\boldsymbol{\pi}_i \odot \boldsymbol{\pi}'_i \right) \right\| + \ell \left| \frac{\pi'_{ij} - \pi_{ij}}{\pi_{ij} \pi'_{ij}} \right| \\
&\leq \frac{\lambda}{\pi_{ij} \min_{k \in [m]} \{\pi_{ik}\}} \left\| \boldsymbol{\beta}_i - \boldsymbol{\beta}'_i \right\| + \frac{\lambda \left\| \boldsymbol{\beta}'_i \right\|}{\min_{k \in [m]} \{\pi_{ik} \pi'_{ik}\}} \left\| \boldsymbol{\pi}'_i - \boldsymbol{\pi}_i \right\| + \ell \frac{1}{\pi_{ij} \pi'_{ij}} \left| \pi'_{ij} - \pi_{ij} \right| \\
&\leq \frac{\lambda}{\pi_{ij} \min_{k \in [m]} \{\pi_{ik}\}} \left\| \boldsymbol{\beta}_i - \boldsymbol{\beta}'_i \right\| + \frac{\lambda \left\| \boldsymbol{\beta}'_i \right\|}{\min_{k \in [m]} \{\pi_{ik} \pi'_{ik}\}} \left\| \boldsymbol{\pi}'_i - \boldsymbol{\pi}_i \right\| + \ell \frac{1}{\pi_{ij} \pi'_{ij}} \left\| \boldsymbol{\pi}'_i - \boldsymbol{\pi}_i \right\| \\
&\leq \frac{\lambda}{\pi^2_{\min}} \left\| \boldsymbol{\beta}_i - \boldsymbol{\beta}'_i \right\| + \frac{\beta'_{\max} \lambda}{\pi^2_{\min}} \left\| \boldsymbol{\pi}'_i - \boldsymbol{\pi}_i \right\| + \ell \frac{1}{\pi^2_{\min}} \left\| \boldsymbol{\pi}'_i - \boldsymbol{\pi}_i \right\| \\
&\leq \max \left(\frac{\lambda}{\pi^2_{\min}}, \frac{\beta'_{\max} \lambda}{\pi^2_{\min}}, \frac{\ell}{\pi^2_{\min}} \right) \left\| (\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) - (\boldsymbol{\beta}'_i, \boldsymbol{\pi}'_i) \right\|
\end{aligned}
$$

Similarly, we also have for all $i \in [n]$, $j \in [m]$ $(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i), (\boldsymbol{\beta}'_i, \boldsymbol{\pi}'_i) \in \mathcal{B}_i$:

$$
\begin{aligned}
&\left| \pi_{ij} \nabla_{x_{ij}} u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i) - \pi'_{ij} \nabla_{x_{ij}} u_i(\boldsymbol{\beta}'_i \oslash \boldsymbol{\pi}'_i) \right| \\
&= \left| \pi_{ij} \left(\nabla_{x_{ij}} u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i) - \nabla_{x_{ij}} u_i(\boldsymbol{\beta}'_i \oslash \boldsymbol{\pi}'_i) \right) - \nabla_{x_{ij}} u_i(\boldsymbol{\beta}'_i \oslash \boldsymbol{\pi}'_i) \left(\pi_{ij} - \pi'_{ij} \right) \right| \\
&\leq \left| \pi_{ij} \left(\nabla_{x_{ij}} u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i) - \nabla_{x_{ij}} u_i(\boldsymbol{\beta}'_i \oslash \boldsymbol{\pi}'_i) \right) \right| + \left| \nabla_{x_{ij}} u_i(\boldsymbol{\beta}'_i \oslash \boldsymbol{\pi}'_i) \left(\pi_{ij} - \pi'_{ij} \right) \right| \\
&= \pi_{ij} \left| \nabla_{x_{ij}} u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i) - \nabla_{x_{ij}} u_i(\boldsymbol{\beta}'_i \oslash \boldsymbol{\pi}'_i) \right| + \left| \nabla_{x_{ij}} u_i(\boldsymbol{\beta}'_i \oslash \boldsymbol{\pi}'_i) \right| \left| \pi_{ij} - \pi'_{ij} \right| \\
&\leq \pi_{ij} \left| \nabla_{x_{ij}} u_i(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i) - \nabla_{x_{ij}} u_i(\boldsymbol{\beta}'_i \oslash \boldsymbol{\pi}'_i) \right| + \ell \left| \pi_{ij} - \pi'_{ij} \right|
\end{aligned}
$$

$$\leq \pi_{ij}\lambda \left|(\boldsymbol{\beta}_i \oslash \boldsymbol{\pi}_i) - (\boldsymbol{\beta}'_i \oslash \boldsymbol{\pi}'_i)\right| - \ell \left|\pi_{ij} - \pi'_{ij}\right|$$
$$\leq \pi_{ij}\lambda \left\|(\boldsymbol{\beta}_i - \boldsymbol{\beta}'_i) \oslash \boldsymbol{\pi}_i - \boldsymbol{\beta}'_i \odot \left(\mathbf{1}_m \oslash \boldsymbol{\pi}_i - \mathbf{1}_m \oslash \boldsymbol{\pi}'_i\right)\right\| + \ell \left|\pi_{ij} - \pi'_{ij}\right|$$
$$\leq \pi_{ij}\lambda \left\|(\boldsymbol{\beta}_i - \boldsymbol{\beta}'_i) \oslash \boldsymbol{\pi}_i\right\| + \pi_{ij}\lambda \left\|\boldsymbol{\beta}'_i \odot \left(\mathbf{1}_m \oslash \boldsymbol{\pi}_i - \mathbf{1}_m \oslash \boldsymbol{\pi}'_i\right)\right\| + \ell \left|\pi_{ij} - \pi'_{ij}\right|$$
$$\leq \frac{\pi_{ij}\lambda}{\min_{k\in[m]} \pi_{ik}} \left\|\boldsymbol{\beta}_i - \boldsymbol{\beta}'_i\right\| + \pi_{ij}\lambda \left\|\boldsymbol{\beta}'_i \odot \left(\mathbf{1}_m \oslash \boldsymbol{\pi}_i - \mathbf{1}_m \oslash \boldsymbol{\pi}'_i\right)\right\| + \ell \left|\pi_{ij} - \pi'_{ij}\right|$$

$$\leq \frac{\pi_{ij}\lambda}{\min_{k\in[m]} \pi_{ik}} \left\|\boldsymbol{\beta}_i - \boldsymbol{\beta}'_i\right\| + \pi_{ij}\lambda \|\boldsymbol{\beta}'_i\| \left\|\mathbf{1}_m \oslash \boldsymbol{\pi}_i - \mathbf{1}_m \oslash \boldsymbol{\pi}'_i\right\| + \ell \left|\pi_{ij} - \pi'_{ij}\right|$$
$$= \frac{\pi_{ij}\lambda}{\min_{k\in[m]} \pi_{ik}} \left\|\boldsymbol{\beta}_i - \boldsymbol{\beta}'_i\right\| + \pi_{ij}\lambda \|\boldsymbol{\beta}'_i\| \left\|\left(\boldsymbol{\pi}_i - \boldsymbol{\pi}'_i\right) \oslash \left(\boldsymbol{\pi}_i \odot \boldsymbol{\pi}'_i\right)\right\| + \ell \left|\pi_{ij} - \pi'_{ij}\right|$$
$$\leq \frac{\pi_{ij}\lambda}{\min_{k\in[m]} \pi_{ik}} \left\|\boldsymbol{\beta}_i - \boldsymbol{\beta}'_i\right\| + \frac{\pi_{ij}\lambda}{\left\{\min_{k\in[m]} \pi_{ik}\pi'_{ik}\right\}} \|\boldsymbol{\beta}'_i\| \left\|\boldsymbol{\pi}_i - \boldsymbol{\pi}'_i\right\| + \ell \left|\pi_{ij} - \pi'_{ij}\right|$$
$$\leq \frac{\pi_{ij}\lambda}{\min_{k\in[m]} \pi_{ik}} \left\|\boldsymbol{\beta}_i - \boldsymbol{\beta}'_i\right\| + \frac{\pi_{ij}\lambda}{\left\{\min_{k\in[m]} \pi_{ik}\pi'_{ik}\right\}} \|\boldsymbol{\beta}'_i\| \left\|\boldsymbol{\pi}_i - \boldsymbol{\pi}'_i\right\| + \ell \left\|\boldsymbol{\pi}_i - \boldsymbol{\pi}'_i\right\|$$
$$\leq \frac{\pi_{\max}\lambda}{\pi_{\min}} \left\|\boldsymbol{\beta}_i - \boldsymbol{\beta}'_i\right\| + \frac{\pi_{\max}\lambda}{\pi_{\min}^2} \|\beta_{\max}\| \left\|\boldsymbol{\pi}_i - \boldsymbol{\pi}'_i\right\| + \ell \left\|\boldsymbol{\pi}_i - \boldsymbol{\pi}'_i\right\|$$
$$\leq \max\left(\frac{\pi_{\max}\lambda}{\pi_{\min}}, \frac{\pi_{\max}\lambda}{\pi_{\min}^2}, \ell\right) \left\|(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) - (\boldsymbol{\beta}'_i, \boldsymbol{\pi}'_i)\right\|$$

Putting it all together, we have for all consumer $i \in [n]$:

$$\left\|\nabla u'_i(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) - \nabla u'_i(\boldsymbol{\beta}'_i, \boldsymbol{\pi}'_i)\right\|$$
$$\leq \sum_{j\in[m]} \left\|\nabla_{\beta_{ij}} u'_i(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) - \nabla_{\beta_{ij}} u'_i(\boldsymbol{\beta}'_i, \boldsymbol{\pi}'_i)\right\| + \sum_{j\in[m]} \left\|\nabla_{\pi_{ij}} u'_i(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) - \nabla_{\pi_{ij}} u'_i(\boldsymbol{\beta}'_i, \boldsymbol{\pi}'_i)\right\|$$
$$\leq \sum_{j\in[m]} \max\left(\frac{\lambda}{\pi_{\min}^2}, \frac{\beta'_{\max}\lambda}{\pi_{\min}^2}, \frac{\ell}{\pi_{\min}^2}\right) \left\|(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) - (\boldsymbol{\beta}'_i, \boldsymbol{\pi}'_i)\right\|$$
$$+ \sum_{j\in[m]} \max\left(\frac{\pi_{\max}\lambda}{\pi_{\min}}, \frac{\pi_{\max}\lambda}{\pi_{\min}^2}, \ell\right) \left\|(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) - (\boldsymbol{\beta}'_i, \boldsymbol{\pi}'_i)\right\|$$
$$= m \max\left(\frac{\lambda}{\pi_{\min}^2}, \frac{\beta'_{\max}\lambda}{\pi_{\min}^2}, \frac{\ell}{\pi_{\min}^2}\right) \left\|(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) - (\boldsymbol{\beta}'_i, \boldsymbol{\pi}'_i)\right\| + m \max\left(\frac{\pi_{\max}\lambda}{\pi_{\min}}, \frac{\pi_{\max}\lambda}{\pi_{\min}^2}, \ell\right) \left\|(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) - (\boldsymbol{\beta}'_i, \boldsymbol{\pi}'_i)\right\|$$
$$= m \left[\max\left(\frac{\lambda}{\pi_{\min}^2}, \frac{\beta'_{\max}\lambda}{\pi_{\min}^2}, \frac{\ell}{\pi_{\min}^2}\right) + \max\left(\frac{\pi_{\max}\lambda}{\pi_{\min}}, \frac{\pi_{\max}\lambda}{\pi_{\min}^2}, \ell\right)\right] \left\|(\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) - (\boldsymbol{\beta}'_i, \boldsymbol{\pi}'_i)\right\|$$

Remark 10.3.4 [Boundedness of bid space]**.**

Recall that in the trading post pseudo-game, the action spaces of the players $i \in [n]$, $\mathcal{B}_i$ are compact by definition. As such, $\pi_{\min}, \pi_{\max}, \beta_{\max}$ are all guaranteed to exist.

With sufficient conditions that ensure the Lipschitz-smoothness of the trading post pseudo-game, combining Lemma 10.3.2, Lemma 10.3.5, Lemma 10.3.6, and Theorem 4.3.1, we have the following theorem.

Theorem 10.3.1 [Convergence of Mirror extragradient Learning Dynamics]**.**

Consider a concave pure exchange economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$, where for all players $i \in [n]$, u_i is ℓ-Lipschitz-continuous and λ-Lipschitz-smooth. Let $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u}')$ be the trading post pseudo-game (Definition 10.2.1) associated with the pure exchange economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$, and h a 1-strongly-convex and κ-Lipschitz-smooth kernel function. Define:

$$\pi_{\min} \doteq \min\{\pi_{ij} : (\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) \in \mathcal{B}_i, \forall i \in [n], j \in [m]\} \ ,$$

$$\pi_{\max} \doteq \max\{\pi_{ij} : (\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) \in \mathcal{B}_i, \forall i \in [n], j \in [m]\} \ ,$$

$$\beta_{\max} \doteq \max\{\beta_{ij} : (\boldsymbol{\beta}_i, \boldsymbol{\pi}_i) \in \mathcal{B}_i, \forall i \in [n], j \in [m]\} \ ,$$

and

$$\lambda' \doteq m\left[\max\left(\frac{\lambda}{\pi_{\min}^2}, \frac{\beta'_{\max}\lambda}{\pi_{\min}^2}, \frac{\ell}{\pi_{\min}^2}\right) + \max\left(\frac{\pi_{\max}\lambda}{\pi_{\min}}, \frac{\pi_{\max}\lambda}{\pi_{\min}^2}, \ell\right)\right] \ .$$

Consider the mirror extratrade dynamics (i.e., the mirror extragradient learning dynamics— Algorithm 7— applied to the trading post pseudo-game) run with the trading post pseudo-game $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u}')$, the kernel function h, a step size $\eta \in \left(0, \frac{1}{\sqrt{2}\lambda'}\right]$, for any time horizon $\tau \in \mathbb{N}$, and outputs $\{(\boldsymbol{\beta}^{(t+0.5)}, \boldsymbol{\pi}^{(t+0.5)}), (\boldsymbol{\beta}^{(t)}, \boldsymbol{\pi}^{(t)})\}_t$. Let $(\boldsymbol{\beta}^{(\tau)}_{\text{best}}, \boldsymbol{\pi}^{(\tau)}_{\text{best}}) \in \arg\min_{(\boldsymbol{\beta}^{(k)}, \boldsymbol{\pi}^{(k)}): k=0,\dots,\tau} \operatorname{div}_h((\boldsymbol{\beta}^{(k+0.5)}, \boldsymbol{\pi}^{(k+0.5)}), (\boldsymbol{\beta}^{(k)}, \boldsymbol{\pi}^{(k)}))$. Then, for some choice of $\tau \in O(\frac{1}{\varepsilon^2})$, $(\boldsymbol{\beta}^{(\tau)}_{\text{best}}, \boldsymbol{\pi}^{(\tau)}_{\text{best}})$ is a ε-first-order VE of $(\mathcal{A}, \boldsymbol{h}, \boldsymbol{u}')$.

In addition, define $\boldsymbol{p}^{(t)} \doteq \sum_{k\in[n]} \boldsymbol{\beta}_k^{(t)}$, and $\boldsymbol{x}_i^{(t)} \doteq \boldsymbol{\beta}_i^{(t)} \oslash \boldsymbol{\pi}_i^{(t)}$, then $\lim_{t\to\infty}(\boldsymbol{x}^{(t)}, \boldsymbol{p}^{(t)}) = (\boldsymbol{x}^*, \boldsymbol{p}^*)$ is an Arrow-Debreu equilibrium of $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$.

10.4 Merit Function Methods for Arrow-Debreu Economies

10.4.1 Merit Functions for Arrow-Debreu Economies

In this section, we investigate the computation of Arrow-Debreu equilibrium beyond pure exchange economies. Unfortunately, as the efficient computation of an Arrow-Debreu equilibrium seems out of reach beyond pure exchange economies, we will loosen our aim of computing an Arrow-Debreu equilibrium to the computation of prices and consumptions which satisfy necessary conditions to be an Arrow-Debreu equilibrium. Our approach to compute such prices and consumptions will be to introduce two separate polynomial-time first- and second-order methods, which we will derive via merit function minimization.

In particular, we will consider the Arrow-Debreu pseudo-game whose set of GNE is equal to the set of Arrow-Debreu competitive equilibria of the associated Arrow-Debreu economy, and then apply the merit function methods for pseudo-games we derived in Section 9.4 to compute an action profile which satisfies necessary condition to be a GNE of the Arrow-Debreu pseudo-game, thus also resulting in prices and consumptions which satisfy necessary conditions to be an Arrow-Debreu equilibrium.

First, let's recall the Arrow-Debreu pseudo-game (Definition 10.2.1) which consists of the following $n+1$ simultaneous optimization problems:

$$\forall i \in [n], \qquad \max_{\boldsymbol{x}_i \in \mathcal{X}'_i : \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p}} u_i(\boldsymbol{x}_i) \qquad \Bigg| \qquad \max_{\boldsymbol{p} \in \Delta_m} \boldsymbol{p} \cdot \left(\sum_{i \in [n]} \boldsymbol{x}_i - \sum_{i \in [n]} \boldsymbol{e}_i \right)$$

Now, as this pseudo-game does not have jointly convex constraints, a VE is not guaranteed to exist in this pseudo-game, as such, none of the methods we derive in Section 9.4 can be applied to it. To overcome this issue, we instead leverage Theorem 9.2.3 which allows us to convert the Arrow-Debreu pseudo-game into a $2n+1$ player game, and apply our merit function methods to solve this game.

Definition 10.4.1 [Arrow-Debreu Game]**.**

Given an Arrow-Debreu economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$, we define the associated $(2n+1)$-player **Arrow-Debreu game** $(n+1, 1, \mathcal{A}, \boldsymbol{u}')$, denoted $(\mathcal{A}, \boldsymbol{u}')$ when clear from context, in which the first n players are called "consumers", the players n through $2n$ are called the "shadow consumers" and the $(2n+1)^{th}$ player is called the "auctioneer" and where:

(Action spaces) For all consumers $i \in [n]$, $\mathcal{A}_i \doteq \mathcal{X}'_i \doteq \left\{ \boldsymbol{x}_i \mid \sum_{k \in [n]} \boldsymbol{x}_k \leq \sum_{k \in [n]} \boldsymbol{e}_k, \forall \boldsymbol{x} \in \mathcal{X} \right\}$,
for all shadow consumers $i \in [n, 2n]$, $\mathcal{A}_i \doteq \Lambda_i \subseteq \mathbb{R}_+$,
and for the auctioneer $\mathcal{A}_{2n+1} \doteq \Delta_m$

(Payoffs) For all consumers $i \in [n]$, $u'_i(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{p}) \doteq u_i(\boldsymbol{x}_i) + \lambda_i (\boldsymbol{e}_i \cdot \boldsymbol{p} - \boldsymbol{x}_i \cdot \boldsymbol{p})$,
for all shadow consumers $i \in [n, 2n]$, $u'_i(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{p}) \doteq -u_i(\boldsymbol{x}_i) - \lambda_i (\boldsymbol{e}_i \cdot \boldsymbol{p} - \boldsymbol{x}_i \cdot \boldsymbol{p})$,
and for the auctioneer, $u'_{n+1}(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{p}) \doteq \boldsymbol{p} \cdot \left(\sum_{i \in [n]} \boldsymbol{x}_i - \sum_{i \in [n]} \boldsymbol{e}_i \right)$.

The Arrow-Debreu game can succinctly be represented by the following $2n + 1$ simultaneous optimization problems:

$$\forall i \in [n], \qquad \max_{\boldsymbol{x}_i \in \mathcal{X}'_i} \min_{\lambda_i \geq 0} u_i(\boldsymbol{x}_i) + \lambda_i (\boldsymbol{e}_i \cdot \boldsymbol{p} - \boldsymbol{x}_i \cdot \boldsymbol{p}) \qquad \Big| \qquad \max_{\boldsymbol{p} \in \Delta_m} \boldsymbol{p} \cdot \left(\sum_{i \in [n]} \boldsymbol{x}_i - \sum_{i \in [n]} \boldsymbol{e}_i \right)$$

We note that a naive choice for the action spaces of the shadow consumers is for all $i \in [n, 2n]$, $\Lambda_i \doteq \mathbb{R}_+$. However, in the results we present in the sequel to ensure polynomial-time convergence of our algorithm, it will be necessary to choose, a non-empty and compact choice of Λ_i to ensure that the payoff functions of the players are Lipschitz-smooth. While we will not delve in the details of how to choose such action spaces to keep our exposition simple and will only present informal expositions of our theorems, we note that under Slater's condition, which is satisfied in quasiconcave Arrow-Debreu economies, it is possible to define such a set. For additional details, we refer the reader to Nedić and Ozdaglar (2009) and Nedic and Ozdaglar (2009), as well as Definition 9.2.6.

Now, notice that in quasiconcave Arrow-Debreu economies, Slater's condition is guaranteed to hold. As such, we have the following corollary of Theorem 9.2.3 and Lemma 10.2.1:

Corollary 10.4.1.

Given a quasiconcave Arrow-Debreu economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$, consider the associated Arrow-Debreu game $(\mathcal{A}, \boldsymbol{u}')$. Then, for any Arrow-Debreu equilibrium $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ of the Arrow-Debreu economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$, there exists $\boldsymbol{\lambda}^* \in \Lambda$ s.t. $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{p}^*)$ is a Nash equilibrium of the Arrow-Debreu game $(\mathcal{A}, \boldsymbol{u}')$.

Conversely, the consumptions and prices $(\boldsymbol{x}^*, \boldsymbol{p}^*)$ of any Nash equilibrium $(\boldsymbol{x}^*, \boldsymbol{\lambda}^*, \boldsymbol{p}^*)$ of the Arrow-Debreu game $(\mathcal{A}, \boldsymbol{u}')$ correspond to an Arrow-Debreu equilibrium of the Arrow-Debreu economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$.

10.4.2 First-Order Market Dynamics for Merit Function Minimization

With this corollary in hand, we can now apply REDA (Algorithm 9) to compute a stationary point of the regularized exploitability associated with the Arrow-Debreu game, or alternatively Algorithm 10 to compute a stationary point of the variational exploitability associated with the Arrow-Debreu game. Our first theorem is a corollary of Theorem 9.4.3, which we present informally to avoid burdening our exposition with highly involved bounds which depend on the Lipschitz-smoothness constant of the utility functions of the consumers, and the radius of the action space of the shadow consumers.

Theorem 10.4.1 [Convergence of REDA in the Arrow-Debreu Game]**.**
Given an Arrow-Debreu economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$, consider the associated Arrow-Debreu game $(\mathcal{A}, \boldsymbol{u}')$. Suppose that for all consumers $i \in [n]$, u_i is Lipschitz-smooth, and for all shadow consumers $i \in [n, 2n]$, the action spaces $\mathcal{A}_i \doteq \Lambda_i$ are non-empty, compact, convex and contain the Nash equilibrium actions of the shadow consumers.

Let $\varepsilon, \alpha > 0$, φ_α be the α-regularized exploitability associated with the Arrow-Debreu game, h be a 1-strongly-convex kernel function, and $\boldsymbol{x}^{(0)}, \boldsymbol{\lambda}^{(0)}, \boldsymbol{p}^{(0)} \in \mathcal{A}$ any initial actions. Then, for some appropriate choice of $\eta > 0$ and $\tau \in O(1/\varepsilon^2)$, the regularized extragradient descent ascent algorithm is guaranteed to compute a ε-stationary point of the α-regularized exploitability φ_α.

10.4.3 Second-Order Market Dynamics for Merit Function Minimization

Similar to Theorem 10.4.1, we also obtain the following corollary of Theorem 9.6.2 for the convergence of the mirror variational learning dynamics when applied to the Arrow-Debreu game.

Theorem 10.4.2.
Given an Arrow-Debreu economy $(\mathcal{X}, \boldsymbol{e}, \boldsymbol{u})$, consider the associated Arrow-Debreu game $(\mathcal{A}, \boldsymbol{u}')$. Suppose that for all consumers $i \in [n]$, u_i is Lipschitz-smooth, and for all shadow consumers $i \in [n, 2n]$, the action spaces $\mathcal{A}_i \doteq \lambda_i$ are non-empty, compact, convex and contain the Nash equilibrium actions of the shadow consumers.

Let $\varepsilon, \alpha > 0$, Ξ_α be the α-variational exploitability associated with the Arrow-Debreu game, h be a 1-strongly-convex kernel function, and $\boldsymbol{x}^{(0)}, \boldsymbol{\lambda}^{(0)}, \boldsymbol{p}^{(0)} \in \mathcal{A}$ any initial actions. Then, for some appropriate choice of

$\eta > 0$ and $\tau \in O(1/\varepsilon)$, the mirror variational learning dynamics is guaranteed to compute a ε-stationary point of the α-variational exploitability Ξ_α.

Part III

Markov Pseudo-Games and Radner Economies

Chapter 11

Scope and Motivation

11.1 Scope

Part III of this thesis is divided into two chapters. In Chapter 12, we[1] introduce our model of Markov pseudo-games (Section 12.1) and define appropriate solution concepts, i.e., equilibria, in Section 12.2, where we establish their existence under suitable conditions. We then present a gradient descent-ascent-based reinforcement learning algorithm (Algorithm 11; TTSSGDA), which provably converges to a solution satisfying the necessary equilibrium conditions via a coupled min-max optimization formulation of the problem (Section 12.3).

In Chapter 13, we apply our theory to Radner economies (or infinite-horizon Markov exchange economies). First, we formulate static exchange economies, i.e., spot markets, as generalized (one-shot) games. Next, we develop infinite-horizon Radner economies by modeling them as instances of our Markov pseudo-games framework. We then invoke our main theorems for Markov pseudo-games to establish the existence of recursive Radner equilibria in Radner economies—the first such result to our knowledge—as well as the convergence of TTSSGDA to this equilibrium. Finally, we present experiments confirming the accuracy and efficiency of TTSSGDA in practice.

11.2 Motivation

In 1896, Léon Walras formulated a mathematical model of markets as a resource allocation system comprising supply and demand functions that map values for resources, called **prices**, to quantities of resources—

[1]The work in Part III was developed in collaboration with Sadie Zhao, Yiling Chen, and Amy Greenwald.

ceteris paribus, i.e., all else being equal. Walras argued that any market would eventually settle into a steady state, which he called **competitive** (nowadays, also called **Walrasian**) **equilibrium**, as a collection of prices and associated supply and demand s.t. the demand is **feasible**, i.e., the demand for each resource is less than or equal to its supply, and **Walras' law** holds, i.e., the value of the supply is equal to the value of the demand. Unlike in Walras' model, real-world markets do not exist in isolation but are part of an **economy**. Indeed, the supply and demand of resources in one market depend not only on prices in that market, but also on the supply and demand of resources in other markets. If every market in an economy is simultaneously at a competitive equilibrium, Walras' law holds for the economy as a whole; this steady state, now a property of the economy, is called a **general equilibrium**.

Beyond Walras' early forays into competitive equilibrium analysis, foremost to the development of the theory of general equilibrium was the introduction of a broad mathematical framework for modeling economies, which is known today as the **Arrow-Debreu competitive economy** (Arrow and Debreu, 1954). In this same paper, Arrow and Debreu developed their seminal game-theoretic model, namely (quasi)concave pseudo-games, and proved the existence of generalized Nash equilibrium in this model. Since this game-theoretic model is sufficiently rich to capture Arrow-Debreu economies, they obtained as a corollary the existence of general equilibrium in these economies.

In their model, Arrow and Debreu posit a set of resources, modeled as commodities, each of which is assigned a price; a set of consumers, each choosing a quantity of each commodity to consume in exchange for their endowment; and a set of firms, each choosing a quantity of each commodity to produce, with prices determining **(aggregate) demand**, i.e., the sum of the consumptions across all consumers, and **(aggregate) supply**, i.e., the sum of endowments and productions across all consumers and firms, respectively. This model is **static**, as it comprises only a single period model, but it is nonetheless rich, as commodities can be state and time contingent, with each one representing a good or service which can be bought or sold in a single time period, but that encodes delivery opportunities at a finite number of distinct points in time. Following Arrow and Debreu's seminal existence result, the literature would slowly turn away from static economies, which do not *explicitly* involve time and uncertainty, such as Arrow-Debreu competitive economies.

Arrow and Debreu's model fails to provide a comprehensive account of the economic activity observed in the real world, especially that which is designed to account for *time* and *uncertainty*. Chief among these activities are **asset markets**, which allow consumers and firms to insure themselves against uncertainty about future states of the world. Indeed, while static economies with state- and time-contingent commodities can *implicitly* incorporate time and uncertainty, the assumption that a complete set of state- and time-contingent commodities are available at the time of trade is highly unrealistic. Arrow (1964) thus proposed to enhance the Arrow-Debreu competitive economy with **assets** (or **securities** or **stocks**),[2] i.e., contracts between two consumers which promise the delivery of commodities by its seller to its buyer at a future date. In particular, Arrow introduced an asset type nowadays known as the ***numéraire* Arrow security**, which transfers one unit of a designated commodity used as a unit of account—**the numéraire**—when a particular state of the world is observed, and nothing otherwise. As the numéraire is often interpreted as money, assets which deliver only some amount of the numéraire, are called **financial assets**.

Formally, Arrow considered a **two-step stochastic exchange economy**. In the initial state, consumers can buy or sell *numéraire* Arrow securities in a **financial asset market**. Following these trades, the economy stochastically transitions to one of finitely many other states in which consumers receive returns on their initial investment and participate in a **spot market**, i.e., a market for the immediate delivery of commodities, modeled as a static exchange economy—which, for our purposes, is better called an **exchange market**.[3] A general equilibrium of this economy is then simply prices for financial assets *and* commodities, which lead to a feasible allocation of all resources (i.e., financial assets and spot market commodities) that satisfies Walras' law.

Arrow (1964) demonstrated that the general equilibrium consumptions of an exchange economy with state- and time-contingent commodities can be implemented by the general equilibrium spot market consumptions of a two-step stochastic economy with a considerably smaller, yet **complete** set of *numéraire* Arrow securities, i.e., a set of securities available for purchase in the first period that allow consumers to

[2]Some authors (e.g., Geanakoplos (1990)) distinguish between assets, stocks, and securities, instead defining securities (resp. stocks) as those assets which are defined exogenously (resp. endogenously), e.g., government bonds (resp. company stocks). As this distinction makes no mathematical difference to our results, and is only relevant to stylized models, we make no such distinction.

[3]An (Arrow-Debreu) exchange economy is simply an (Arrow-Debreu) competitive economy without firms. Historically, for simplicity, it has become standard practice *not* to model firms, as most, if not all, results extend directly to settings with firms. In line with this practice, we do not model firms, but note that our results and methods also extend directly to settings that include firms.

transfer wealth to *all* possible states of the world that can be realized in the second period. In conjunction with the welfare theorems (Debreu, 1951a; Arrow, 1951b), this result implies that economies with **complete financial asset markets**, i.e., economies with such a complete set of securities, achieve a Pareto-optimal allocation of commodities by ensuring optimal risk-bearing via financial asset markets; and conversely, any Pareto-optimal allocation of commodities in economies with time and uncertainty can be realized as a competitive equilibrium of a complete financial asset market.

Arrow's contributions led to the development of a new class of general equilibrium models, namely **stochastic economies** (or **dynamic stochastic general equilibrium—DSGE—models**) (Geanakoplos, 1990).[4] At a high-level, these models comprise a sequence of world states and spot markets, which are linked across time by asset markets, with each next state of the world (resp. spot market) determined by a stochastic process that is independent of market interactions (resp. dependent only on their asset purchase) in the current state. Mathematically, the key difference between a static and a stochastic economy is that consumers in a stochastic economy face a collection of budget constraints, one per time-step, rather than only one. Indeed, Arrow (1964)'s proof that general equilibrium consumptions in stochastic complete economies are equivalent to general equilibrium consumptions in static state- and time-contingent commodity economies relies on proving that the many budget constraints in a complete stochastic economy can be reduced to a single one.

Stochastic economies were introduced to model arbitrary finite time horizons (Radner, 1968) and a variety of risky asset classes (e.g., stocks (Diamond, 1967), risky assets (Lintner, 1975), derivatives (Black and Scholes, 1973), capital assets (Mossin, 1966), debts (Modigliani and Miller, 1958) etc.), eventually leading to the emergence of **stochastic economies with incomplete asset markets** (Magill and Shafer, 1991; Magill and Quinzii, 2002; Geanakoplos, 1990), or colloquially, **(incomplete) stochastic economies**.[5] Unlike in Arrow's stochastic economy, the asset market is not complete in such economies, so consumers cannot necessarily insure themselves against all future world states.

[4]As these models incorporate both time and uncertainty, they are often referred to as dynamic stochastic general equilibrium models. Nonetheless, we opt for the stochastic economy nomenclature, because, as we demonstrate in this paper, these economies can be seen as instances of (generalized) stochastic games.

[5]While many authors have called these models incomplete economies (Geanakoplos, 1990; Magill and Quinzii, 2002; Magill and Shafer, 1991), these models capture both incomplete and complete asset markets. In contrast, we refer to stochastic economies with incomplete or complete asset markets as **stochastic economies**, adding the (in)complete epithet only when necessary to indicate that the asset market is (in)complete.

The archetypal stochastic economy is the **Radner stochastic exchange economy**, deriving its name from Radner's proof of existence of a general equilibrium in his model (Radner, 1972). Radner's economy is a finite-horizon stochastic economy comprising a sequence of spot markets, linked across time by asset markets. At each time period, a finite set of consumers observe a world state and trade in an asset market and a spot market, modeled as an exchange market. Each **asset market** comprises **assets**, modelled as time-contingent **generalized Arrow securities**, which specify quantities of the commodities the seller is obliged to transfer to its buyer, should the relevant state of the economy be realized at some specified future time.[6] Consumers can buy and sell assets, thereby transferring their wealth across time, all the while insuring themselves against uncertainty about the future. The canonical solution concept for stochastic economies, **Radner equilibrium** (also called **sequential competitive equilibrium**[7] (Mas-Colell et al., 1995), **rational expectations equilibrium** (Radner, 1979), and **general equilibrium with incomplete markets** (Geanakoplos, 1990)), is a collection of history-dependent prices for commodities and assets, as well as history-dependent consumptions of commodities and portfolios of assets, such that, for all histories, the aggregate demand for commodities and the **aggregate demand for assets** (i.e., the total number of assets bought) are feasible and satisfy Walras' law.

In spite of substantial interest in stochastic economies among microeconomists throughout the 1970s, the literature eventually trailed off, perhaps due to a seeming difficulty in proving existence of a general equilibrium in simple economies with incomplete asset markets that allow assets to be sold short (Geanakoplos, 1990), or to the lack of a second welfare theorem (Dreze, 1974; Hart, 1975). Financial and macroeconomists stepped up, however, with financial economists seeking to further develop the theoretical aspects of stochastic economies (see, for instance, Magill and Quinzii (2002)), and macroeconomists seeking practical methods by which to solve stochastic economies in order to determine the impact of various policy choices (via simulation; see, for instance, Sargent and Ljungqvist (2000)).

Radner economies are one of the new and interesting directions in this more recent work on stochastic economies. Infinite horizon models come with one significant difficulty that has no counterpart in a finite

[6]Here, Arrow securities are "generalized" in the sense that they can deliver different quantities of *many* commodities at different states of the world, rather than only one unit of a commodity at only one state of the world. Although Arrow (1964) considered only *numéraire* securities, his theory was subsequently generalized to generalized Arrow securities (Geanakoplos, 1990).

[7]This terminology does not contradict the economy being at a competitive equilibrium, but rather indicates that at all times, the spot and asset markets are at a competitive equilibrium, hence implying the overall economy is at a general equilibrium.

horizon model, namely the possibility for agents to run a **Ponzi scheme** via asset markets, in which they borrow but then indefinitely postpone repaying their debts by refinancing them continually, from one period to the next. From this perspective infinite horizon models represent very interesting objects of study, not only theoretically; it has also been argued that they are a better modeling paradigm for macroeconomists who employ simulations (Magill and Quinzii, 1994), because they facilitate the modeling of complex phenomena, such as asset bubbles (Huang and Werner, 2000), which can be impacted by economic policy decisions.

Magill and Quinzii (1994) introduced an extension of Radner's model to an infinite horizon setting, albeit with financial assets, and presented suitable assumptions under which a sequential competitive equilibrium is guaranteed to exist in this model. Progress on the computational aspects of stochastic economies has been slow, however, and mostly confined to finite horizon settings (see, Sargent and Ljungqvist (2000) and Volume 2 of Taylor and Woodford (1999) for a standard survey and Fernández-Villaverde (2023) for a more recent entry-level survey of computational methods used by macroeconomists). Indeed, demands for novel computational methods for solving macroeconomic models, and theoretical frameworks in which to understand their computational complexity, have been repeatedly shared by macroeconomists (Taylor and Woodford, 1999). This gap in the literature points to a novel research opportunity; however, it is challenging for non-macroeconomists to approach these problems with their computational tools.

11.3 Contributions

In Chapter 12, we introduce Markov pseudo-games, and we establish the existence of (pure) **generalized Markov perfect equilibria (GMPE)** in concave Markov pseudo-games (Theorem 12.2.1). This result can be seen as a stochastic generalization of Arrow and Debreu (1954)'s existence result for (pure) generalized Nash equilibrium in concave pseudo-games (Facchinei and Kanzow, 2010a). To the best of our knowledge, it also implies the existence of pure (or deterministic) Markov perfect equilibria in a large class of continuous-action Markov games for which existence was heretofore known only in mixed (or randomized) policies (Fink, 1964; Takahashi, 1964).

Although the computation of GMPE is PPAD-hard in general, because GMPE generalize Nash equilibrium, we reduce this computational problem to generative adversarial learning between a generator, who produces a candidate equilibrium policy profile, and an adversary, who produces a policy profile of best responses to the candidate equilibrium (Goktas et al., 2023a) (Observation 12.3.1). Assuming parameterized policies, and taking advantage of the recent progress on solving generative adversarial learning problems (e.g., (Lin et al., 2020; Daskalakis et al., 2020a)), we show that a policy profile which is a stationary point of the exploitability (i.e., the players' cumulative maximum regret) can be computed in polynomial time under mild assumptions (Theorem 12.3.1). This result implies that a policy profile that satisfies necessary first-order stationarity conditions for a GMPE in Markov pseudo-games with a bounded best-response mismatch coefficient (Lemma 12.3.4)—i.e., those Markov pseudo-games in which states explored by any GMPE are easily explored under the initial state distribution—can be computed in polynomial time, a result which is analogous known computational results for zero-sum Markov games (Daskalakis et al., 2020a). As our theoretical computational guarantees apply to policies represented by neural networks, we obtain the first, to our knowledge, deep reinforcement learning algorithm with theoretical guarantees for general-sum games.

In Chapter 13, we introduce an extension of Magill and Quinzii (1994)'s infinite horizon exchange economy, which we call the **Radner economy**. On the one hand, our model generalizes Magill and Quinzii's to a setting with arbitrary, not just financial assets; on the other hand, we restrict the transition model to be Markov. The Markov restriction allows us to prove the existence of a **recursive Radner equilibrium (RRE)** (Mehra and Prescott, 1977) (Theorem 13.1.1). Our proof reformulates the set of RRE of any Radner economy as the set of GMPE of an associated generalized Markov game (Theorem 13.1.1). To our knowledge, ours is the first result of its kind for such a general setting, as previous recursive competitive equilibrium existence proofs were restricted to economies with one consumer (also called the representative agent), one commodity, or one asset (Mehra and Prescott, 1977; Prescott and Mehra, 1980). The aforementioned results allow us to conclude that a stationary point of the exploitability of the Markov pseudo-game associated with any Radner economy can be computed in polynomial time (Theorem 13.1.2).

Finally, in Section 13.2 we implement our policy gradient method in the form of a generative adversarial policy network (GAPNet), and use it to try to find RRE in three Radner economies with three different types of utility functions. Experimentally, we find that our GAPNet produces approximate equilibrium policies that are closer to GMPE than those produced by a standard baseline for solving stochastic economies.

Chapter 12

Markov Pseudo-Games

12.1 Background

12.1.1 Mathematical Background

Throughout, we adopt the following notational shorthand: If a function $f : \mathcal{X} \to \mathbb{R}$ is Lipschitz-continuous (resp. Lipschitz-smooth), for simplicity, we will denote its Lipschitz-continuity (resp. Lipschitz-smoothness) constant by $\ell_f \geq 0$ (resp. $\ell_{\nabla f} \geq 0$).

We will also require notions of stochastic convexity related to stochastic dominance of probability measures (Atakan, 2003b).

Definition 12.1.1 [Stochastic Convexity/Concavity]**.**
Given non-empty and convex parameter and outcome spaces $\mathcal{W}$ and $\mathcal{O}$ respectively, a conditional probability distribution $\boldsymbol{w} \mapsto \rho(\cdot \mid \boldsymbol{w}) \in \Delta(\mathcal{O})$ is said to be **stochastically convex** (resp. **stochastically concave**) in $\boldsymbol{w} \in \mathcal{W}$ if for all continuous, bounded, and convex (resp. concave) functions $v : \mathcal{O} \to \mathbb{R}$, $\lambda \in (0, 1)$, and $\boldsymbol{w}', \boldsymbol{w}^\dagger \in \mathcal{W}$ s.t. $\overline{\boldsymbol{w}} = \lambda \boldsymbol{w}' + (1 - \lambda)\boldsymbol{w}^\dagger$, it holds that $\mathbb{E}_{O \sim \rho(\cdot|\overline{\boldsymbol{w}})} [v(O)] \leq$ (resp. $\geq$) $\lambda \, \mathbb{E}_{O \sim \rho(\cdot|\boldsymbol{w}')} [v(O)] + (1 - \lambda) \, \mathbb{E}_{O \sim \rho(\cdot|\boldsymbol{w}^\dagger)} [v(O)]$.

12.1.2 Markov Pseudo-Games

We begin by developing our formal game model. The games we study are stochastic, in the sense of Shapley (1953), Fink (1964), and Takahashi (1964). Further, they are pseudo-games, in the sense of Arrow and Debreu (1954). Arrow and Debreu introduced pseudo-games to establish the existence of competitive equilibrium in their seminal model of an exchange economy, where an auctioneer sets prices that determine the consumers' budget sets, and hence their feasible consumptions. It is this dependency among the players' feasible actions that characterizes pseudo-games. We model stochastic pseudo-games, and dub them **Markov pseudo-games**, as the games are Markov in that the stochastic transitions depend only on the most recent state and player actions.

An **(infinite horizon discounted) Markov pseudo-game** $\mathcal{M} \doteq (n, m, \mathcal{S}, \mathcal{A}, \mathcal{X}, \boldsymbol{r}, \rho, \gamma, \mu)$ is an n-player dynamic game played over an infinite discrete time horizon. The game starts at time $t = 0$ in some initial state $S^{(0)} \sim \mu \in \Delta(\mathcal{S})$ drawn randomly from a set of states $\mathcal{S} \subseteq \mathbb{R}^l$. At this and each subsequent time period $t = 1, 2, \ldots$, the players encounter a state $\boldsymbol{s}^{(t)} \in \mathcal{S}$, in which each $i \in [n]$ simultaneously takes an **action** $\boldsymbol{a}_i^{(t)} \in \mathcal{X}_i(\boldsymbol{s}^{(t)}, \boldsymbol{a}_{-i}^{(t)})$ from a **set of feasible actions** $\mathcal{X}_i(\boldsymbol{s}^{(t)}, \boldsymbol{a}_{-i}^{(t)}) \subseteq \mathcal{A}_i \subseteq \mathbb{R}^m$, determined by a **feasible action correspondence** $\mathcal{X}_i : \mathcal{S} \times \mathcal{A}_{-i} \rightrightarrows \mathcal{A}_i$, which takes as input the current state $\boldsymbol{s}^{(t)}$ and the other players' actions $\boldsymbol{a}_{-i}^{(t)} \in \mathcal{A}_{-i}$, and outputs a subset of the ith player's action space $\mathcal{A}_i$. We define $\mathcal{X}(\boldsymbol{s}, \boldsymbol{a}) \doteq \bigtimes_{i \in [n]} \mathcal{X}_i(\boldsymbol{s}, \boldsymbol{a}_{-i})$.

Once the players have taken their actions $\boldsymbol{a}^{(t)} \doteq (\boldsymbol{a}_1^{(t)}, \ldots, \boldsymbol{a}_n^{(t)})$, each player $i \in [n]$ receives a **reward** $r_i(\boldsymbol{s}^{(t)}, \boldsymbol{a}^{(t)})$ given by a **reward function** $\boldsymbol{r} : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^n$, after which the game either ends with probability $1 - \gamma$, where $\gamma \in (0, 1)$ is called the **discount factor**,[1] or continues on to time period $t + 1$, transitioning to a new state $S^{(t+1)} \sim \rho(\cdot \mid \boldsymbol{s}^{(t)}, \boldsymbol{a}^{(t)})$, according to a **transition** probability function $\rho : \mathcal{S} \times \mathcal{S} \times \mathcal{A} \to [0, 1]$, where $\rho(\boldsymbol{s}^{(t+1)} \mid \boldsymbol{s}^{(t)}, \boldsymbol{a}^{(t)}) \in [0, 1]$ denotes the probability of transitioning to state $\boldsymbol{s}^{(t+1)} \in \mathcal{S}$ from state $\boldsymbol{s}^{(t)} \in \mathcal{S}$ when action profile $\boldsymbol{a}^{(t)} \in \mathcal{A}$ is played.

Our focus is on continuous-state and continuous-action Markov pseudo-games, where the state and action spaces are non-empty and compact, and the reward functions are continuous and bounded in each of $\boldsymbol{s}$ and $\boldsymbol{a}$, holding the other fixed.

A **history** $\boldsymbol{h} \in \mathcal{H}^\tau \doteq (\mathcal{S} \times \mathcal{A})^\tau \times \mathcal{S}$ of length $\tau \in \mathbb{N}$ is a sequence of states and action profiles $\boldsymbol{h} = ((\boldsymbol{s}^{(t)}, \boldsymbol{a}^{(t)})_{t=0}^{\tau-1}, \boldsymbol{s}^{(\tau)})$ s.t. a history of length 0 corresponds only to the initial state of the game. For any history $\boldsymbol{h} = ((\boldsymbol{s}^{(t)}, \boldsymbol{a}^{(t)})_{t=0}^{\tau-1}, \boldsymbol{s}^{(\tau)})$ of length $\tau \in \mathbb{N}$, we denote by $\boldsymbol{h}_{:\tau'}$ the first $\tau' \in [0 : \tau]$ steps of $\boldsymbol{h}$, i.e., $\boldsymbol{h}_{:\tau'} = ((\boldsymbol{s}^{(t)}, \boldsymbol{a}^{(t)})_{t=0}^{\tau'-1}, \boldsymbol{s}^{(\tau')})$.

[1] Our results generalize to settings with per-player discount factors $\gamma_i \in (0, 1)$, where the discount rates express the players' intertemporal preferences over game outcomes at each time-step. .

Overloading notation, we define the **history space** $\mathcal{H} \doteq \bigcup_{\tau=0}^{\infty} \mathcal{H}^{\tau}$. For any player $i \in [n]$, a **policy** $\boldsymbol{\pi}_i : \mathcal{H} \to \mathcal{A}_i$ is a mapping from histories of any length to i's space of (pure) actions. We define the space of all (deterministic) policies as $\mathcal{P}_i \doteq \{\boldsymbol{\pi}_i : \mathcal{H} \to \mathcal{A}_i\}$.[2] A **Markov policy** (Maskin and Tirole, 2001) $\boldsymbol{\pi}_i$ is a policy s.t. $\boldsymbol{\pi}_i(\boldsymbol{s}^{(\tau)}) = \boldsymbol{\pi}_i(\boldsymbol{h}_{:\tau})$, for all histories $\boldsymbol{h} \in \mathcal{H}^{\tau}$ of length $\tau \in \mathbb{N}_+$, where $\boldsymbol{s}^{(\tau)}$ denotes the final state of history $\boldsymbol{h}$. As Markov policies are only state-contingent, we can compactly represent the space of all Markov policies for player $i \in [n]$ as $\mathcal{P}_i^{\text{markov}} \doteq \{\boldsymbol{\pi}_i : \mathcal{S} \to \mathcal{A}_i\}$.

Fixing player $i \in [n]$ and $\boldsymbol{\pi}_{-i} \in \mathcal{P}_{-i}$, given history $\boldsymbol{h} \in \mathcal{H}^{\tau}$, we define the **feasible policy correspondence**

$$\mathcal{F}_i(\boldsymbol{\pi}_{-i}) \doteq \{\boldsymbol{\pi}_i \in \mathcal{P}_i \mid \forall \boldsymbol{h} \in \mathcal{H}, \boldsymbol{\pi}_i(\boldsymbol{h}) \in \mathcal{X}_i(\boldsymbol{s}^{(\tau)}, \boldsymbol{\pi}_{-i}(\boldsymbol{h}))\},$$

and for any $\mathcal{P}^{\text{sub}} \subseteq \mathcal{P}^{\text{markov}}$, the **feasible subclass policy correspondence**

$$\mathcal{F}_i^{\text{sub}}(\boldsymbol{\pi}_{-i}) \doteq \{\boldsymbol{\pi}_i \in \mathcal{P}_i^{\text{sub}} \mid \forall \boldsymbol{s} \in \mathcal{S}, \boldsymbol{\pi}_i(\boldsymbol{s}) \in \mathcal{X}_i(\boldsymbol{s}, \boldsymbol{\pi}_{-i}(\boldsymbol{s}))\}.$$

Of particular interest is $\mathcal{F}_i^{\text{markov}}(\boldsymbol{\pi}_{-i})$ itself, obtained when $\mathcal{P}^{\text{sub}} = \mathcal{P}^{\text{markov}}$.

Given a policy profile $\boldsymbol{\pi} \in \mathcal{P}$ and a history $\boldsymbol{h} \in \mathcal{H}^{\tau}$, we define the **discounted history distribution** assuming initial state distribution μ as

$$\nu_{\mu}^{\boldsymbol{\pi},\tau}(\boldsymbol{h}) = \mu(\boldsymbol{s}^{(0)}) \prod_{t=0}^{\tau-1} \gamma^t \rho(\boldsymbol{s}^{(t+1)} \mid \boldsymbol{s}^{(t)}, \boldsymbol{a}^{(t)}) \mathbb{1}_{\{\boldsymbol{\pi}(\boldsymbol{h}_{:t})\}}(\boldsymbol{a}^{(t)}).$$

Overloading notation, we also define **the set of all realizable trajectories** $\mathcal{H}^{\boldsymbol{\pi}}$ of length τ under policy $\boldsymbol{\pi}$ as $\mathcal{H}^{\boldsymbol{\pi}} \doteq \operatorname{supp}(\nu_{\mu}^{\boldsymbol{\pi},\tau})$, i.e., the set of all histories that occur with non-zero probability. We then denote by $\nu_{\mu}^{\boldsymbol{\pi}} \doteq \nu_{\mu}^{\boldsymbol{\pi},\infty}$, and by $H = \left(S^{(0)}, (A^{(t)}, S^{(t+1)})_{t=0}^{\tau-1}\right)$ any randomly sampled history from $\nu_{\mu}^{\boldsymbol{\pi},\tau}$. Finally, we define the **discounted state-visitation distribution**, again assuming initial state distribution μ, as

$$\delta_{\mu}^{\boldsymbol{\pi}}(\boldsymbol{s}) = \sum_{\tau=0}^{\infty} \int_{\boldsymbol{h} \in \mathcal{H}^{\boldsymbol{\pi}} : \boldsymbol{s}^{(\tau)} = \boldsymbol{s}} \nu_{\mu}^{\boldsymbol{\pi},\tau}(\boldsymbol{h}).$$

For any policy profile $\boldsymbol{\pi} \in \mathcal{P}$, the **state-value function** $\boldsymbol{v}^{\boldsymbol{\pi}} : \mathcal{S} \to \mathbb{R}^n$ and the **action-value function** $\boldsymbol{q}^{\boldsymbol{\pi}} : \mathcal{S} \times \mathcal{A} \to \mathbb{R}^n$ are defined, respectively, as

$$\boldsymbol{v}^{\boldsymbol{\pi}}(\boldsymbol{s}) \doteq \mathop{\mathbb{E}}_{S^{(t+1)} \sim \rho(\cdot \mid S^{(t)}, A^{(t)})} \left[\sum_{t=0}^{\infty} \boldsymbol{r}(S^{(t)}, A^{(t)}) \mid S^{(0)} = \boldsymbol{s}, A^{(t)} = \boldsymbol{\pi}(S^{(t)}) \right] \tag{12.1}$$

$$\boldsymbol{q}^{\boldsymbol{\pi}}(\boldsymbol{s}, \boldsymbol{a}) \doteq \mathop{\mathbb{E}}_{S^{(t+1)} \sim \rho(\cdot \mid S^{(t)}, A^{(t)})} \left[\sum_{t=0}^{\infty} \boldsymbol{r}(S^{(t)}, A^{(t)}) \mid S^{(0)} = \boldsymbol{s}, A^{(0)} = \boldsymbol{a}, A^{(t+1)} = \boldsymbol{\pi}(S^{(t+1)}) \right] . \tag{12.2}$$

[2] A **mixed policy** is simply a distribution over pure policies, i.e., an element of $\Delta(\mathcal{P}_i)$. Moreover, any mixed policy can be equivalently represented as a mapping $\boldsymbol{\pi}_i^{\text{mixed}} : \mathcal{H} \to \Delta(\mathcal{A}_i)$ from histories to distributions over actions s.t. at any history $\boldsymbol{h} \in \mathcal{H}$, player i plays action $\boldsymbol{a}_i \sim \boldsymbol{\pi}_i(\boldsymbol{h})$. An analogous definition extends directly to mixed Markov policies as well.

Overloading notation, for any arbitrary initial state distribution $\upsilon \in \Delta(\mathcal{S})$ and policy profile $\boldsymbol{\pi}$, we denote by $\boldsymbol{v}^{\boldsymbol{\pi}}(\upsilon) \doteq \mathbb{E}_{S \sim \upsilon}[\boldsymbol{v}^{\boldsymbol{\pi}}(S)]$.

Finally, the **(expected cumulative) payoff** associated with policy profile $\boldsymbol{\pi} \in \mathcal{P}$ is given by $\boldsymbol{u}(\boldsymbol{\pi}) \doteq \boldsymbol{v}^{\boldsymbol{\pi}}(\mu)$.

12.2 Solution Concepts and Existence

Having defined our model, we now define two natural solution concepts, and establish their existence. Our first solution concept is based on the usual notion of Nash equilibrium (1950b), yet applied to Markov pseudo-games. Our second is based on the notion of subgame-perfect equilibrium in extensive-form games, a strengthening of Nash equilibrium with the additional requirement that an equilibrium be Nash not just at the start of the game, but at all states encountered during play. In the context of stochastic games, such equilibria are called "recursive," or "Markov perfect." Following Bellman (1966) and Arrow and Debreu (1954), we identify natural assumptions that guarantee the existence of equilibrium in (pure) Markov policies, meaning deterministic policies that depend only on the current state, not on the history. When applied to Radner economies, this theorem implies existence of (pure) recursive Radner equilibrium, to our knowledge the first result of its kind.

Definition 12.2.1 [Approximate Generalized Nash Equilibrium]**.**
An ε-**generalized Nash equilibrium (**ε**-GNE)** $\boldsymbol{\pi}^* \in \mathcal{F}(\boldsymbol{\pi}^*)$ is a policy profile s.t. for all states $\boldsymbol{s} \in \mathcal{S}$ and players $i \in [n]$,

$$u_i(\boldsymbol{\pi}^*) \geq \max_{\boldsymbol{\pi}_i \in \mathcal{F}_i(\boldsymbol{\pi}^*_{-i})} u_i(\boldsymbol{\pi}_i, \boldsymbol{\pi}^*_{-i}) - \varepsilon \ .$$

We call a 0-GNE simply a GNE.

Definition 12.2.2 [Approximate Generalized Markov Perfect Equilibrium]**.**
An ε-**generalized Markov perfect equilibrium (**ε**-GMPE)** $\boldsymbol{\pi}^* \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^*)$ is a Markov policy profile s.t. for all states $\boldsymbol{s} \in \mathcal{S}$ and players $i \in [n]$,

$$v_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}) \geq \max_{\boldsymbol{\pi}_i \in \mathcal{F}_i(\boldsymbol{\pi}^*_{-i})} v_i^{(\boldsymbol{\pi}_i, \boldsymbol{\pi}^*_{-i})}(\boldsymbol{s}) - \varepsilon \ .$$

We call a 0-GMPE simply a GMPE.

As GMPE is a stronger notion than GNE, every ε-GMPE is an ε-GNE.

Our existence result is based on two assumptions, one set of assumptions on the game, and another set of assumptions on the policy subclass for which existence is sought. The following assumption on the Markov pseudo-game ensures that the Markov pseudo-game is "continuous" and "concave".

Assumption 12.2.1 [Concave Markov Pseudo-game]**.**

For all players $i \in [n]$, assume

1. $\mathcal{A}_i$ is convex
2. $\mathcal{X}_i(\boldsymbol{s}, \cdot)$ is upper- and lower-hemicontinuous, for all $\boldsymbol{s} \in \mathcal{S}$
3. $\mathcal{X}_i(\boldsymbol{s}, \boldsymbol{a}_{-i})$ is non-empty, convex, and compact, for all $\boldsymbol{s} \in \mathcal{S}$ and $\boldsymbol{a}_{-i} \in \mathcal{A}_{-i}$
4. For any policy $\boldsymbol{\pi} \in \mathcal{P}$, $\boldsymbol{a}_i \mapsto q_i^{\boldsymbol{\pi}}(\boldsymbol{s}, \boldsymbol{a}_i, \boldsymbol{a}_{-i})$ is continuous and concave over $\mathcal{X}_i(\boldsymbol{s}, \boldsymbol{a}_{-i})$, for all $\boldsymbol{s} \in \mathcal{S}$ and $\boldsymbol{a}_{-i} \in \mathcal{A}_{-i}$

The following assumption of the policy subclass ensures that the policy subclass is expressive and well-behaved enough for it to represent a GMPE. We note that the set of Markov policies by definition satisfies the following assumption.

Assumption 12.2.2 [Policy Class]**.**

Given $\mathcal{P}^{\text{sub}} \subseteq \mathcal{P}^{\text{markov}}$, assume:

1. $\mathcal{P}^{\text{sub}}$ is non-empty, compact, and convex
2. (Closure under policy improvement): For each $\boldsymbol{\pi} \in \mathcal{P}^{\text{sub}}$, there exists $\boldsymbol{\pi}^+ \in \mathcal{P}^{\text{sub}}$ s.t. $q_i^{\boldsymbol{\pi}}(\boldsymbol{s}, \boldsymbol{\pi}_i^+(\boldsymbol{s}), \boldsymbol{\pi}_{-i}(\boldsymbol{s})) = \max_{\boldsymbol{\pi}_i' \in \mathcal{F}(\boldsymbol{\pi}_{-i})} q_i^{\boldsymbol{\pi}}(\boldsymbol{s}, \boldsymbol{\pi}_i'(\boldsymbol{s}), \boldsymbol{\pi}_{-i}(\boldsymbol{s}))$, for all $i \in [n]$ and $\boldsymbol{s} \in \mathcal{S}$

Assumption 2, introduced as Condition 1 in Bhandari and Russo (2019), ensures that the policy class under consideration (e.g., $\mathcal{P}^{\text{sub}} \subseteq \mathcal{P}^{\text{markov}}$) is expressive enough to include best responses. With the above assumptions in hand, we can prove the existence of a GMPE using the Kakutani-Glicksberg fixed point theorem (Glicksberg, 1952) (Theorem 2.4.1, Chapter 2)

Theorem 12.2.1.

Let $\mathcal{M}$ be a Markov pseudo-game for which Assumption 12.2.1 holds, and let $\mathcal{P}^{\text{sub}} \subseteq \mathcal{P}^{\text{markov}}$ be a subspace of Markov policy profiles that satisfies Assumption 12.2.2. Then, there exists a policy $\boldsymbol{\pi}^* \in \mathcal{P}^{\text{sub}}$ such that $\boldsymbol{\pi}^*$ is an GMPE of $\mathcal{M}$.

With solution concepts and their existence, we next turn our attention to computation.

12.3 Merit Function Minimization for Generalized Markov Perfect Equilibrium

Our approach to computing a GMPE in a Markov pseudo-game $\mathcal{M}$ is to minimize a **merit function** associated with $\mathcal{M}$, i.e., a function whose minima coincides with the pseudo-game's GMPE. Our choice of merit function, a common one in game theory, is **exploitability** $\varphi : \mathcal{P} \to \mathbb{R}_+$, defined as $\varphi(\boldsymbol{\pi}) \doteq \sum_{i\in[n]} \left[\max_{\boldsymbol{\pi}'_i \in \mathcal{F}_i^{\text{markov}}(\boldsymbol{\pi}_{-i})} u_i(\boldsymbol{\pi}'_i, \boldsymbol{\pi}_{-i}) - u_i(\boldsymbol{\pi})\right]$. In words, exploitability is the sum of the players' maximal unilateral payoff deviations.

Exploitability, however, is a merit function for GNE, *not* GMPE; **state exploitability**, $\phi(\boldsymbol{s}, \boldsymbol{\pi}) = \sum_{i\in[n]}[\max_{\boldsymbol{\pi}'_i \in \mathcal{F}_i^{\text{markov}}(\boldsymbol{\pi}_{-i})} v_i^{(\boldsymbol{\pi}'_i, \boldsymbol{\pi}_{-i})}(\boldsymbol{s}) - v_i^{\boldsymbol{\pi}}(\boldsymbol{s})]$ at all states $\boldsymbol{s} \in \mathcal{S}$, is a merit function for GMPE. Nevertheless, as we show in the sequel, for a large class of Markov pseudo-games, namely those with a bounded best-response mismatch coefficient (see Section 12.3.3), the set of Markov policies that minimize exploitability equals the set of GMPE, making our approach a sensible one.

We are not out of the woods yet, however, as exploitability is non-convex in general, even in one-shot finite games (Nash, 1950a). Although Markov pseudo-games can afford a convex exploitability (see, for instance (Flam and Ruszczynski, 1994)), it is unlikely that all do, as GNE computation is PPAD-hard (Chen et al., 2009; Daskalakis et al., 2009). Accordingly, we instead set our sights on computing a **stationary point** of the exploitability, i.e., a policy profile $\boldsymbol{\pi}^* \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^*)$ s.t. for any other policy $\boldsymbol{\pi} \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^*)$, it holds that $\min_{\boldsymbol{h} \in \mathcal{D}\varphi(\boldsymbol{\pi}^*)} \langle \boldsymbol{h}, \boldsymbol{\pi}^* - \boldsymbol{\pi} \rangle \leq 0$.[3] Such a point satisfies the necessary conditions of a GMPE.

In this paper, we study Markov pseudo-games with possibly continuous state and action spaces. As such, we can only hope to compute an *approximate* stationary point of the exploitability in finite time. Defining a notion of approximate stationarity for exploitability is, however, a challenge, because exploitability is non-differentiable in general (once again, even in one-shot finite games).

Given an approximation parameter $\varepsilon \geq 0$, a natural definition of an ε-stationary point might be a policy profile $\boldsymbol{\pi}^* \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^*)$ s.t. for any other policy $\boldsymbol{\pi} \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^*)$, it holds that $\min_{\boldsymbol{h} \in \mathcal{D}\varphi(\boldsymbol{\pi}^*)} \langle \boldsymbol{h}, \boldsymbol{\pi}^* - \boldsymbol{\pi} \rangle \leq \varepsilon$.

[3] While we provide a definition of a(n approximate) stationary point for expositional purposes at present, an observant reader might have noticed the exploitability φ is a mapping from a function space to the positive reals, and its Frêchet (sub)derivative is ill-specified without a clear definition of the normed vector space of policies on which exploitability is defined. Further, even when clearly specified, such a (sub)derivative might not exist. The precise meaning of a derivative of the exploitability and its stationary points will be introduced more rigorously once we have suitably parameterized the policy spaces.

Exploitability is not necessarily Lipschitz-smooth, however, so in general it may not be possible to compute an ε-stationary point in $\text{poly}(1/\varepsilon)$ evaluations of the (sub)gradient of the exploitability.[4]

To address, this challenge, a common approach in the optimization literature (see, for instance Appendix H, Definition 19 of Liu et al. (2021)) is to consider an alternative definition known as (ε, δ)-stationarity. Given approximation parameters $\varepsilon, \delta \geq 0$, an (ε, δ)-stationary point of the exploitability is a policy profile $\boldsymbol{\pi}^* \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^*)$ for which there exists a δ-close policy $\boldsymbol{\pi}^\dagger \in \mathcal{P}$ with $\|\boldsymbol{\pi}^\dagger - \boldsymbol{\pi}^*\| \leq \delta$ s.t. for any other policy $\boldsymbol{\pi} \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^\dagger)$, it holds that $\min_{\boldsymbol{h} \in \mathcal{D}\varphi(\boldsymbol{\pi}^\dagger)} \langle \boldsymbol{h}, \boldsymbol{\pi}^\dagger - \boldsymbol{\pi} \rangle \leq \varepsilon$. The exploitability minimization method we introduce can compute such an approximate stationary point in polynomial time. Furthermore, asymptotically, our method is guaranteed to converge to an exact stationary point of the exploitability.

More precisely, following Goktas and Greenwald (2022), who minimize exploitability to solve for variational equilibria in (one-shot) pseudo-games, we first formulate our problem as the quasi-optimization problem of minimizing exploitability,[5] and then transform this problem into a coupled min-max optimization (i.e., a two-player zero-sum game) whose objective is cumulative regret, rather than the potentially ill-behaved exploitability. Under suitable parametrization, such problems are amenable to polynomial-time solutions via simultaneous gradient descent ascent (Arrow et al., 1958), assuming the objective is Lipschitz smooth in both players' decision variables and gradient dominated in the inner player's. We thus formulate the requisite assumptions to ensure these properties hold of cumulative regret in our game, which in turn allows us to show that **two time scale simultaneous stochastic gradient descent ascent (TTSSGDA)** converges to an $(\varepsilon, O(\varepsilon))$-stationary point of the exploitability in $\text{poly}(1/\varepsilon)$ gradient steps.

12.3.1 Exploitability Minimization

Given a Markov pseudo-game $\mathcal{M}$ and two policy profiles $\boldsymbol{\pi}, \boldsymbol{\pi}' \in \mathcal{P}$, we define the **state cumulative regret** at state $\boldsymbol{s} \in \mathcal{S}$ as $\psi(\boldsymbol{s}, \boldsymbol{\pi}, \boldsymbol{\pi}') = \sum_{i \in [n]} \left[v_i^{(\boldsymbol{\pi}'_i, \boldsymbol{\pi}_{-i})}(\boldsymbol{s}) - v_i^{\boldsymbol{\pi}}(\boldsymbol{s}) \right]$; the **expected cumulative regret** for

[4] To see this, consider the convex minimization problem $\min_{x \in \mathbb{R}} f(x) = |x|$. The minimum of this optimization occurs at $x = 0$, which is a stationary point since a (sub)derivative of f at $x = 0$ is 0. However, for $x < 0$, we have $\frac{\partial f(x)}{\partial x} = -1$, and for $x > 0$, we have $\frac{\partial f(x)}{\partial x} = 1$. Hence, any $x \in \mathbb{R} \setminus \{0\}$ can at best be a 1-stationary point, i.e., $\left|\frac{\partial f(x)}{\partial x}\right| = 1$. Hence, for this optimization problem, it is not even possible to guarantee the existence of an ε-stationary point distinct from $x = 0$, assuming $\varepsilon \in (0, 1)$, let alone the computation of an ε-stationary point x^* s.t. $\left|\frac{\partial f(x^*)}{\partial x}\right| \leq \varepsilon$.

[5] Here, "quasi" refers to the fact that a solution to this problem is both a minimizer of exploitability and a fixed point of an operator, such as $\mathcal{F}$ or $\mathcal{F}^{\text{markov}}$.

any initial state distribution $\upsilon \in \Delta(\mathcal{S})$ as $\psi(\upsilon, \boldsymbol{\pi}, \boldsymbol{\pi}') = \mathbb{E}_{\boldsymbol{s} \sim \upsilon}[\psi(\boldsymbol{s}, \boldsymbol{\pi}, \boldsymbol{\pi}')]$, and the **cumulative regret** as $\Psi(\boldsymbol{\pi}, \boldsymbol{\pi}') = \psi(\mu, \boldsymbol{\pi}, \boldsymbol{\pi}')$. Additionally, we define the **state exploitability** of a policy profile $\boldsymbol{\pi}$ at state $\boldsymbol{s} \in \mathcal{S}$ as $\phi(\boldsymbol{s}, \boldsymbol{\pi}) = \sum_{i \in [n]} \max_{\boldsymbol{\pi}'_i \in \mathcal{F}_i^{\text{markov}}(\boldsymbol{\pi}_{-i})} v_i^{(\boldsymbol{\pi}'_i, \boldsymbol{\pi}_{-i})}(\boldsymbol{s}) - v_i^{\boldsymbol{\pi}}(\boldsymbol{s})$; , the **expected exploitability** of a policy profile $\boldsymbol{\pi}$ for any initial state distribution $\upsilon \in \Delta(\mathcal{S})$ as $\phi(\upsilon, \boldsymbol{\pi}) = \mathbb{E}_{\boldsymbol{s} \sim \upsilon}[\phi(\boldsymbol{s}, \boldsymbol{\pi})]$, and **exploitability** as $\varphi(\boldsymbol{\pi}) = \sum_{i \in [n]} \max_{\boldsymbol{\pi}'_i \in \mathcal{F}_i^{\text{markov}}(\boldsymbol{\pi}_{-i})} u_i(\boldsymbol{\pi}'_i, \boldsymbol{\pi}_{-i})$.

In the above, we restrict our attention to the subclass $\mathcal{P}^{\text{markov}} \subseteq \mathcal{P}$ of (pure) Markov policies. This restriction is without loss of generality, because finding an optimal policy that maximizes a state-value or payoff function, while the other players' policies remain fixed, reduces to solving a Markov decision process (MDP), and an optimal (possibly history-dependent) policy in an MDP is guaranteed to exist in the space of (pure) Markov policies under very mild continuity and compactness assumptions (Puterman, 2014). Indeed, the next lemma justifies this restriction.

Lemma 12.3.1.

Given a Markov pseudo-game $\mathcal{M}$ for which Assumption 12.2.1 holds, a Markov policy profile $\boldsymbol{\pi}^* \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^*)$ is a GMPE if and only if $\phi(\boldsymbol{s}, \boldsymbol{\pi}^*) = 0$, for all states $\boldsymbol{s} \in \mathcal{S}$. Similarly, a policy profile $\boldsymbol{\pi}^* \in \mathcal{F}(\boldsymbol{\pi}^*)$ is an GNE if and only if $\varphi(\boldsymbol{\pi}^*) = 0$.

This lemma tells us that we can reformulate the problem of computing a GMPE as the quasi-minimization problem of minimizing state exploitability, i.e., $\min_{\boldsymbol{\pi} \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi})} \phi(\boldsymbol{s}, \boldsymbol{\pi})$, at all states $\boldsymbol{s} \in \mathcal{S}$ simultaneously. The same is true of computing a GNE and exploitability.

This straightforward reformulation of GMPE (resp. GNE) in terms of state exploitability (resp. exploitability) does not immediately lend itself to computation, as exploitability minimization is non-trivial, because exploitability is neither convex nor differentiable in general. Following Goktas and Greenwald (2022), we can reformulate these problems yet again, this time as coupled quasi-min-max optimization problems (Wald, 1945). We proceed to do so now; however, we restrict our attention to exploitability, and hence GNE, knowing that we will later show that minimizing exploitability suffices to minimize state exploitability, and thereby find GMPE.

Observation 12.3.1.

Given a Markov pseudo-game $\mathcal{M}$,

$$\min_{\boldsymbol{\pi} \in \mathcal{F}(\boldsymbol{\pi})} \varphi(\boldsymbol{\pi}) = \min_{\boldsymbol{\pi} \in \mathcal{F}(\boldsymbol{\pi})} \max_{\boldsymbol{\pi}' \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi})} \Psi(\boldsymbol{\pi}, \boldsymbol{\pi}') \ . \tag{12.3}$$

While the above observation makes progress towards our goal of reformulating exploitability minimization in a tractable manner, the problem remains challenging to solve for two reasons: first, a fixed point computation is required to solve the outer player's minimization problem; second, the inner player's policy space depends on the choice of outer policy. We overcome these difficulties by choosing suitable policy parameterizations.

12.3.2 Policy Parameterization

In a coupled min-max optimization problem, any solution to the inner player's maximization problem is implicitly parameterized by the outer player's decision. We restructure the jointly feasible Markov policy class to represent this dependence explicitly.

Define the class of **dependent policies** $\mathcal{R} \doteq \{\boldsymbol{\rho} : \mathcal{S} \times \mathcal{A} \to \mathcal{A} \mid \forall (\boldsymbol{s}, \boldsymbol{a}) \in \mathcal{S} \times \mathcal{A},\ \boldsymbol{\rho}(\boldsymbol{s}, \boldsymbol{a}) \in \mathcal{X}(\boldsymbol{s}, \boldsymbol{a})\} = \bigtimes_{i \in [n]} \{\boldsymbol{\rho}_i : \mathcal{S} \times \mathcal{A}_i \to \mathcal{A}_{-i} \mid \forall (\boldsymbol{s}, \boldsymbol{a}_{-i}) \in \mathcal{S} \times \mathcal{A}_{-i},\ \boldsymbol{\rho}_i(\boldsymbol{s}, \boldsymbol{a}_{-i}) \in \mathcal{X}_i(\boldsymbol{s}, \boldsymbol{a}_{-i})\}$. With this definition in hand we arrive at an *uncoupled* quasi-min-max optimization problem:

Lemma 12.3.2.

Given a Markov pseudo-game $\mathcal{M}$,

$$\min_{\boldsymbol{\pi} \in \mathcal{F}(\boldsymbol{\pi})} \max_{\boldsymbol{\pi}' \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi})} \Psi(\boldsymbol{\pi}, \boldsymbol{\pi}') = \min_{\boldsymbol{\pi} \in \mathcal{F}(\boldsymbol{\pi})} \max_{\boldsymbol{\rho} \in \mathcal{R}} \Psi(\boldsymbol{\pi}, \boldsymbol{\rho}(\cdot, \boldsymbol{\pi}(\cdot))) \ . \tag{12.4}$$

It can be expensive to represent the aforementioned dependence in policies explicitly. This situation can be naturally rectified, however, by a suitable policy parameterization. A suitable policy parameterization can also allow us to represent the set of fixed points s.t. $\boldsymbol{\pi} \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi})$ more efficiently in practice (Goktas et al., 2023a).

Define a **parameterization scheme** $(\boldsymbol{\pi}, \boldsymbol{\rho}, \mathbb{R}^\Omega, \mathbb{R}^\Sigma)$ as comprising a unconstrained parameter space $\mathbb{R}^\Omega$ and parametric policy profile function $\boldsymbol{\pi} : \mathcal{S} \times \mathbb{R}^\Omega \to \mathcal{A}$ for the outer player, and an unconstrained parameter space $\mathbb{R}^\Sigma$ and parametric policy profile function $\boldsymbol{\rho} : \mathcal{S} \times \mathcal{A} \times \mathbb{R}^\Sigma \to \mathcal{A}$ for the inner player.

Given such a scheme, we restrict the players' policies to be parameterized: i.e., the outer player's space of policies $\mathcal{P}^{\mathbb{R}^\Omega} = \{\boldsymbol{\pi} : \mathcal{S} \times \mathbb{R}^\Omega \to \mathcal{A} \mid \boldsymbol{\omega} \in \mathbb{R}^\Omega\} \subseteq \mathcal{P}^{\text{markov}}$, while the inner player's space of policies $\mathcal{R}^{\mathbb{R}^\Sigma} = \{\boldsymbol{\rho} : \mathcal{S} \times \mathcal{A} \times \mathbb{R}^\Sigma \to \mathcal{A} \mid \boldsymbol{\sigma} \in \mathbb{R}^\Sigma\}$. Using these parametrization, we redefine $\boldsymbol{v}^{\boldsymbol{\omega}} \doteq \boldsymbol{v}^{\boldsymbol{\pi}(\cdot;\boldsymbol{\omega})}$, $\boldsymbol{q}^{\boldsymbol{\omega}} \doteq \boldsymbol{q}^{\boldsymbol{\pi}(\cdot;\boldsymbol{\omega})}$, $\boldsymbol{u}(\boldsymbol{\omega}) = \boldsymbol{u}(\boldsymbol{\pi}(\cdot;\boldsymbol{\omega}))$, and $\nu_\mu^{\boldsymbol{\omega}} = \nu_\mu^{\boldsymbol{\pi}(\cdot;\boldsymbol{\omega})}$; $\boldsymbol{v}^{\boldsymbol{\sigma}(\boldsymbol{\omega})} \doteq \boldsymbol{v}^{\boldsymbol{\rho}(\cdot,\boldsymbol{\pi}(\cdot;\boldsymbol{\omega});\boldsymbol{\sigma})}$; $\boldsymbol{q}^{\boldsymbol{\sigma}(\boldsymbol{\omega})} \doteq \boldsymbol{q}^{\boldsymbol{\rho}(\cdot,\boldsymbol{\pi}(\cdot;\boldsymbol{\omega});\boldsymbol{\sigma})}$; $\boldsymbol{u}(\boldsymbol{\sigma}(\boldsymbol{\omega})) = \boldsymbol{u}(\boldsymbol{\rho}(\cdot,\boldsymbol{\pi}(\cdot;\boldsymbol{\omega});\boldsymbol{\sigma}))$; $\nu_\mu^{\boldsymbol{\sigma}(\boldsymbol{\omega})} = \nu_\mu^{\boldsymbol{\rho}(\cdot,\boldsymbol{\pi}(\cdot;\boldsymbol{\omega});\boldsymbol{\sigma})}$. With these definitions in place, we make the following assumption on our parametrization.

Assumption 12.3.1 [Parametrization for Min-Max Optimization]**.**
Given a Markov pseudo-game $\mathcal{M}$ and a parameterization scheme $(\boldsymbol{\pi}, \boldsymbol{\rho}, \mathbb{R}^\Omega, \mathbb{R}^\Sigma)$, assume:

1. for all $\boldsymbol{\omega} \in \mathbb{R}^\Omega$, $\boldsymbol{\pi}(\boldsymbol{s};\boldsymbol{\omega}) \in \mathcal{X}(\boldsymbol{s}, \boldsymbol{\pi}(\boldsymbol{s};\boldsymbol{\omega}))$, for all $\boldsymbol{s} \in \mathcal{S}$
2. for all $\boldsymbol{\sigma} \in \mathbb{R}^\Sigma$, $\boldsymbol{\rho}(\boldsymbol{s}, \boldsymbol{a};\boldsymbol{\sigma}) \in \mathcal{X}(\boldsymbol{s}, \boldsymbol{a})$, for all $(\boldsymbol{s}, \boldsymbol{a}) \in \mathcal{S} \times \mathcal{A}$

Assuming a policy parameterization scheme that satisfies Assumption 12.3.1, we restate our goal, state exploitability minimization, one last time as the following min-max optimization problem:

$$\min_{\boldsymbol{\omega} \in \mathbb{R}^\Omega} \max_{\boldsymbol{\sigma} \in \mathbb{R}^\Sigma} \Psi(\boldsymbol{\omega}, \boldsymbol{\sigma}) \doteq \Psi(\boldsymbol{\pi}(\cdot;\boldsymbol{\omega}), \boldsymbol{\rho}(\cdot, \boldsymbol{\pi}(\cdot;\boldsymbol{\omega});\boldsymbol{\sigma})) \ . \tag{12.5}$$

Now, given unconstrained parameter space, we are able to simplify our definition of approximate stationary point and obtain our target definition.

Definition 12.3.1.
Given $\varepsilon, \delta \geq 0$, a (ε, δ)**-stationary point of the exploitability** is a policy parameter $\boldsymbol{\omega}^* \in \mathbb{R}^\Omega$ for which there exists a δ-close policy parameter $\boldsymbol{\omega}^\dagger \in \mathbb{R}^\Omega$ with $\|\boldsymbol{\omega}^* - \boldsymbol{\omega}^\dagger\| \leq \delta$ s.t. $\min_{\boldsymbol{h} \in \mathcal{D}\varphi(\boldsymbol{\omega}^\dagger)} \|\boldsymbol{h}\| \leq \varepsilon$.

12.3.3 State Exploitability Minimization

Returning to our stated objective, namely *state* exploitability minimization, we turn our attention to obtaining a tractable characterization of this goal. Specifically, we argue that it suffices to minimize exploitability, rather than state exploitability, as any policy profile that is a stationary point of exploitability is also a stationary point of state exploitability across all states simultaneously, under suitable assumptions.

Our first lemma states that a stationary point of the exploitability is almost surely also a stationary point of the state exploitability at all states. Moreover, if the initial state distribution has full support, then any

(ε, δ)-stationary point of the exploitability can be converted into an $(\varepsilon/\alpha, \delta)$-stationary point of the *state* exploitability, with probability at least $1 - \alpha$.

Lemma 12.3.3.

Given a Markov pseudo-game $\mathcal{M}$, for $\boldsymbol{\omega} \in \mathbb{R}^{\Omega}$, suppose that $\phi(\boldsymbol{s}, \cdot)$ is differentiable at $\boldsymbol{\omega}$ for all $\boldsymbol{s} \in \mathcal{S}$. If $\|\nabla_{\boldsymbol{\omega}} \varphi(\boldsymbol{\omega})\| = 0$, then, for all states $\boldsymbol{s} \in \mathcal{S}$, $\|\nabla_{\boldsymbol{\omega}} \phi(\boldsymbol{s}, \boldsymbol{\omega})\| = 0$ μ-almost surely, i.e., $\mu(\{\boldsymbol{s} \in \mathcal{S} \mid \|\nabla_{\boldsymbol{\omega}} \phi(\boldsymbol{s}, \boldsymbol{\omega})\| = 0\}) = 1$. Moreover, for any $\varepsilon > 0$ and $\delta \in [0, 1]$, if $\operatorname{supp}(\mu) = \mathcal{S}$ and $\|\nabla_{\boldsymbol{\omega}} \varphi(\boldsymbol{\omega})\| \leq \varepsilon$, then with probability at least $1 - \delta$, $\|\nabla_{\boldsymbol{\omega}} \phi(\boldsymbol{s}, \boldsymbol{\omega})\| \leq \varepsilon/\delta$.

In fact, we can strengthen this probabilistic equivalence to a deterministic one by restricting our attention to Markov pseudo-games with bounded best-response mismatch coefficients. Our best-response mismatch coefficient generalizes the minimax mismatch coefficient in two-player settings (Daskalakis et al., 2020a) and the distribution mismatch coefficient in single-agent settings (Agarwal et al., 2020).

Definition 12.3.2 [Best-Response Mismatch Coefficient]**.**

Given $\mathcal{M}$ with initial state distribution μ and alternative state distribution $\upsilon \in \Delta(\mathcal{S})$, and letting $\Phi_i(\boldsymbol{\pi}_{-i}) \doteq \arg\max_{\boldsymbol{\pi}'_i \in \mathcal{F}_i^{\text{markov}}(\boldsymbol{\pi}_{-i})} u_i(\boldsymbol{\pi}'_i, \boldsymbol{\pi}_{-i})$ denote the set of best response policies for player i when the other players play policy profile $\boldsymbol{\pi}_{-i}$, we define the **best-response mismatch coefficient** for policy profile $\boldsymbol{\pi}$ as

$$C_{br}(\boldsymbol{\pi}, \mu, \upsilon) \doteq \max_{i \in [n]} \max_{\boldsymbol{\pi}'_i \in \Phi_i(\boldsymbol{\pi}_{-i})} \left(\frac{1}{1-\gamma}\right)^2 \left\| \frac{\delta_\upsilon^{(\boldsymbol{\pi}'_i, \boldsymbol{\pi}_{-i})}}{\mu} \right\|_\infty \left\| \frac{\delta_\upsilon^{\boldsymbol{\pi}}}{\mu} \right\|_\infty .$$

Lemma 12.3.4.

Let $\mathcal{M}$ be a Markov pseudo-game with initial state distribution μ. Given policy parameter $\boldsymbol{\omega} \in \mathbb{R}^{\Omega}$ and arbitrary state distribution $\upsilon \in \Delta(\mathcal{S})$, suppose that both $\phi(\mu, \cdot)$ and $\phi(\upsilon, \cdot)$ are differentiable at $\boldsymbol{\omega}$, then we have: $\|\nabla \phi(\upsilon, \boldsymbol{\omega})\| \leq C_{br}(\boldsymbol{\pi}(\cdot; \boldsymbol{\omega}^*), \mu, \upsilon) \|\nabla \varphi(\boldsymbol{\omega})\|$.

Once again, Lemma 12.3.3 states that any approximate stationary point of exploitability is also an approximate stationary point of state exploitability with high probability, while Lemma 12.3.4 upper bounds state exploitability in terms of exploitability, when the best-response mismatch coefficient is bounded. Together, these two lemmas imply that finding a policy profile this is a stationary point of exploitability is sufficient for find a policy profile this is a stationary point of state exploitability, and hence one that satisfies the necessary conditions of a GMPE.

12.3.4 Algorithmic Assumptions

We are nearly ready to describe our reinforcement learning algorithm for computing a stationary point of Equation (12.5), and thereby finding a policy profile that satisfies the necessary conditions of a GMPE. As Equation (12.5) is a two-player zero-sum game, our method is a variant of simultaneous gradient descent ascent (GDA) (Arrow et al., 1958), meaning it adjusts its parameters based on first-order information until it reaches a (first-order) stationary point. Polynomial-time convergence of GDA typically requires that the objective be Lipschitz smooth in both decision variables, and gradient dominated in the inner one, which in our application, translates to the cumulative regret $\Psi(\boldsymbol{\omega}, \boldsymbol{\sigma})$ being Lipschitz smooth in $(\boldsymbol{\omega}, \boldsymbol{\sigma})$ and gradient dominated in $\boldsymbol{\sigma}$. These conditions are ensured, under the following assumptions on the Markov pseudo-game.

Assumption 12.3.2 [Lipschitz Smooth Payoffs]**.**
Given a Markov pseudo-game $\mathcal{M}$ and a parameterization scheme $(\boldsymbol{\pi}, \boldsymbol{\rho}, \mathbb{R}^\Omega, \mathbb{R}^\Sigma)$, assume:

1. $\mathbb{R}^\Omega$ and $\mathbb{R}^\Sigma$ are non-empty, compact, and convex
2. $\boldsymbol{\omega} \mapsto \boldsymbol{\pi}(\boldsymbol{s}; \boldsymbol{\omega})$ is twice continuously differentiable, for all $\boldsymbol{s} \in \mathcal{S}$, and $\boldsymbol{\sigma} \mapsto \boldsymbol{\rho}(\boldsymbol{s}, \boldsymbol{a}; \boldsymbol{\sigma})$ is twice continuously differentiable, for all $(\boldsymbol{s}, \boldsymbol{a}) \in \mathcal{S} \times \mathcal{A}$
3. $\boldsymbol{a} \mapsto \boldsymbol{r}(\boldsymbol{s}, \boldsymbol{a})$ is twice continuously differentiable, for all $\boldsymbol{s} \in \mathcal{S}$
4. $\boldsymbol{a} \mapsto \rho(\boldsymbol{s}' \mid \boldsymbol{s}, \boldsymbol{a})$ is twice continuously differentiable, for all $\boldsymbol{s}, \boldsymbol{s}' \in \mathcal{S}$.

Assumption 12.3.3 [Gradient Dominance Conditions]**.**
Given a Markov pseudo-game $\mathcal{M}$ together with a parameterization scheme $(\boldsymbol{\pi}, \boldsymbol{\rho}, \mathbb{R}^\Omega, \mathbb{R}^\Sigma)$, assume:

1. (Closure under policy improvement): For each $\boldsymbol{\omega} \in \mathbb{R}^\Omega$, there exists $\boldsymbol{\sigma} \in \mathbb{R}^\Sigma$ s.t. $q_i^{\boldsymbol{\omega}}(\boldsymbol{s}, \boldsymbol{\rho}_i(\boldsymbol{s}, \boldsymbol{\pi}(\boldsymbol{s}; \boldsymbol{\omega}); \boldsymbol{\sigma}), \boldsymbol{\pi}_{-i}(\boldsymbol{s}; \boldsymbol{\omega})) = \max_{\boldsymbol{\pi}'_i \in \mathcal{F}_i(\boldsymbol{\pi}(\cdot; \boldsymbol{\omega}))} q_i^{\boldsymbol{\omega}}(\boldsymbol{s}, \boldsymbol{\pi}'_i(\boldsymbol{s}), \boldsymbol{\pi}_{-i}(\boldsymbol{s}; \boldsymbol{\omega}))$ for all $i \in [n]$, $\boldsymbol{s} \in \mathcal{S}$.
2. (Concavity of cumulative regret) $\boldsymbol{\sigma} \mapsto q_i^{\boldsymbol{\omega}'}(\boldsymbol{s}, \boldsymbol{\rho}_i(\boldsymbol{s}, \boldsymbol{\pi}_{-i}(\boldsymbol{s}; \boldsymbol{\omega}); \boldsymbol{\sigma}), \boldsymbol{\pi}_{-i}(\boldsymbol{s}; \boldsymbol{\omega}))$ is concave, for all $\boldsymbol{s} \in \mathcal{S}$ and $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathbb{R}^\Omega$.

Algorithm 11 Two time-scale simultaneous SGDA (TTSSGDA)

Inputs: $\mathcal{M}, (\boldsymbol{\pi}, \boldsymbol{\rho}, \mathbb{R}^{\Omega}, \mathbb{R}^{\Sigma}), \eta_{\boldsymbol{\omega}}, \eta_{\boldsymbol{\sigma}}, \boldsymbol{\omega}^{(0)}, \boldsymbol{\sigma}^{(0)}, T$

Outputs: $(\boldsymbol{\omega}^{(t)}, \boldsymbol{\sigma}^{(t)})_{t=0}^{T}$

1: Build gradient estimator $\widehat{G}$ associated with $\mathcal{M}$
2: **for** $t = 0, \ldots, T-1$ **do**
3: $\quad \boldsymbol{h} \sim \nu^{\boldsymbol{\omega}}, \boldsymbol{h}' \sim \bigtimes_{i\in[n]} \nu^{(\boldsymbol{\sigma}_i(\boldsymbol{\omega}_{-i}), \boldsymbol{\omega}_{-i})}$
4: $\quad \boldsymbol{\omega}^{(t+1)} \leftarrow \boldsymbol{\omega}^{(t)} - \eta_{\boldsymbol{\omega}} \widehat{G_{\boldsymbol{\omega}}}(\boldsymbol{\omega}^{(t)}, \boldsymbol{\sigma}^{(t)}; \boldsymbol{h}, \boldsymbol{h}')$
5: $\quad \boldsymbol{\sigma}^{(t+1)} \leftarrow \boldsymbol{\sigma}^{(t)} + \eta_{\boldsymbol{\sigma}} \widehat{G_{\boldsymbol{\sigma}}}(\boldsymbol{\omega}^{(t)}, \boldsymbol{\sigma}^{(t)}; \boldsymbol{h}, \boldsymbol{h}')$
6: **return** $(\boldsymbol{\omega}^{(t)}, \boldsymbol{\sigma}^{(t)})_{t=0}^{T}$

12.3.5 Algorithm and Convergence

Finally, we present our algorithm for finding an approximate stationary point of exploitability, and thus state exploitability. The algorithm we use is two time-scale stochastic simultaneous gradient descent-ascent (TTSSGDA), first analyzed by Lin et al. (2020); Daskalakis et al. (2020a), for which we prove best-iterate convergence to an $(\varepsilon, O(\varepsilon))$-stationary point of exploitability after taking $\mathrm{poly}(1/\varepsilon)$ gradient steps under Assumptions 12.3.2 and 12.3.3.

Recall that Assumption 12.3.2 guarantees Lipschitz smoothness w.r.t. to both $\boldsymbol{\omega}$ and $\boldsymbol{\sigma}$, while Assumption 12.3.3 guarantees gradient dominance w.r.t $\boldsymbol{\sigma}$. As the gradient of cumulative regret involves an expectation over histories, we assume that we can simulate trajectories of play $\boldsymbol{h} \sim \nu_{\mu}^{\boldsymbol{\pi}}$ according to the history distribution $\nu_{\mu}^{\boldsymbol{\pi}}$, for any policy profile $\boldsymbol{\pi}$, and that doing so provides both value and gradient information for the rewards and transition probabilities along simulated trajectories. That is, we rely on a differentiable game simulator (see, for instance Suh et al. (2022)), meaning a stochastic first-order oracle that returns the gradients of the rewards $\boldsymbol{r}$ and transition probabilities ρ, which we query to estimate deviation payoffs, and ultimately cumulative regrets.

Under this assumption, we estimate these values using realized trajectories from the history distribution $\boldsymbol{h} \sim \nu_{\mu}^{\boldsymbol{\omega}}$ induced by the outer player's policy, and the deviation history distribution $\boldsymbol{h}^{\boldsymbol{\sigma}} \sim \bigtimes_{i\in[n]} \nu_{\mu}^{(\boldsymbol{\sigma}_i(\boldsymbol{\omega}_{-i}), \boldsymbol{\omega}_{-i})}$

induced by the inner player's policy. More specifically, for all policies $\boldsymbol{\pi} \in \mathcal{P}^{\mathrm{markov}}$ and histories $\boldsymbol{h} \in \mathcal{H}^{\tau}$, the **payoff estimator** for player $i \in [n]$ is given by:

$$\widehat{u}_i(\boldsymbol{\pi}; \boldsymbol{h}) \doteq \sum_{t=0}^{\tau-1} \mu(\boldsymbol{s}^{(0)}) r_i(\boldsymbol{s}^{(t)}, \boldsymbol{\pi}'(\boldsymbol{s}^{(t)})) \prod_{k=0}^{t-1} \gamma^k \rho(\boldsymbol{s}^{(k+1)} \mid \boldsymbol{s}^{(k)}, (\boldsymbol{s}^{(k)})) \ .$$

Furthermore, for all $\boldsymbol{\omega} \in \mathbb{R}^{\Omega}$, $\boldsymbol{\sigma} \in \mathbb{R}^{\Sigma}$, $\boldsymbol{h} \sim \nu_{\mu}^{\boldsymbol{\omega}}$, and $\boldsymbol{h}^{\boldsymbol{\sigma}} = (\boldsymbol{h}_1^{\boldsymbol{\sigma}}, \cdots, \boldsymbol{h}_n^{\boldsymbol{\sigma}}) \sim \times_{i \in [n]} \nu_{\mu}^{(\boldsymbol{\sigma}_i(\boldsymbol{\omega}_{-i}), \boldsymbol{\omega}_i)}$, the **cumulative regret estimator** is given by $\widehat{\Psi}(\boldsymbol{\omega}, \boldsymbol{\sigma}; \boldsymbol{h}, \boldsymbol{h}') \doteq \sum_{i \in [n]} \widehat{u}_i(\boldsymbol{\rho}_i(\ \cdot\ , \boldsymbol{\pi}_{-i}(\cdot; \boldsymbol{\omega}); \boldsymbol{\sigma}), \boldsymbol{\pi}_{-i}(\ \cdot\ , \boldsymbol{\omega}); \boldsymbol{h}_i^{\boldsymbol{\sigma}}) - \widehat{u}_i(\boldsymbol{\pi}(\cdot; \boldsymbol{\omega}); \boldsymbol{h})$, while the **cumulative regret gradient estimator** is given by $\widehat{G}(\boldsymbol{\omega}, \boldsymbol{\sigma}; \boldsymbol{h}, \boldsymbol{h}^{\boldsymbol{\sigma}}) \doteq (\nabla_{\boldsymbol{\omega}} \widehat{\Psi}(\boldsymbol{\omega}, \boldsymbol{\sigma}; \boldsymbol{h}, \boldsymbol{h}'), \nabla_{\boldsymbol{\sigma}} \widehat{\Psi}(\boldsymbol{\omega}, \boldsymbol{\sigma}; \boldsymbol{h}, \boldsymbol{h}^{\boldsymbol{\sigma}}))$.

Our main theorem requires one final definition, namely the **equilibrium distribution mismatch coefficient** $\left\| \frac{\partial \delta_{\mu}^{\boldsymbol{\pi}^*}}{\partial \mu} \right\|_{\infty}$, defined as the Radon-Nikodym derivative of the state-visitation distribution of the GNE $\boldsymbol{\pi}^*$ w.r.t. the initial state distribution μ. This coefficient, which measures the inherent difficulty of visiting states under the equilibrium policy $\boldsymbol{\pi}^*$—without knowing $\boldsymbol{\pi}^*$—is closely related to other distribution mismatch coefficients used in the analysis of policy gradient methods (Agarwal et al., 2020).

We now state our main theorem, namely that, under the assumptions outlined above, Algorithm 11 computes values for the policy parameters that nearly satisfy the necessary conditions for an MGPNE in polynomially many gradient steps, or equivalently, calls to the differentiable simulator.

Theorem 12.3.1.

Given a Markov pseudo-game $\mathcal{M}$, and a parameterization scheme $(\boldsymbol{\pi}, \boldsymbol{\rho}, \mathbb{R}^{\Omega}, \mathbb{R}^{\Sigma})$, suppose Assumption 12.2.1, 12.3.2, and 12.3.3 hold. For any $\delta > 0$, set $\varepsilon = \delta \| C_{br}(\cdot, \mu, \cdot) \|_{\infty}^{-1}$. If Algorithm 11 is run with inputs that satisfy, $\eta_{\boldsymbol{\omega}}, \eta_{\boldsymbol{\sigma}} \asymp \mathrm{poly}(\varepsilon, \| \partial \delta_{\mu}^{\boldsymbol{\pi}^*} / \partial \mu \|_{\infty}, \frac{1}{1-\gamma}, \ell_{\nabla \Psi}^{-1}, \ell_{\Psi}^{-1})$, then for some $T \in \mathrm{poly}\left(\varepsilon^{-1}, (1-\gamma)^{-1}, \| \partial \delta_{\mu}^{\boldsymbol{\pi}^*} / \partial \mu \|_{\infty}, \ell_{\nabla \Psi}, \ell_{\Psi}, \mathrm{diam}(\mathbb{R}^{\Omega} \times \mathbb{R}^{\Sigma}), \eta_{\boldsymbol{\omega}}^{-1}\right)$, there exists $\boldsymbol{\omega}_{\mathrm{best}}^{(T)} = \boldsymbol{\omega}^{(k)}$ with $k \leq T$ that is a $(\varepsilon, \varepsilon/2\ell_{\Psi})$-stationary point of the exploitability, i.e., there exists $\boldsymbol{\omega}^* \in \mathbb{R}^{\Omega}$ s.t. $\| \boldsymbol{\omega}_{\mathrm{best}}^{(T)} - \boldsymbol{\omega}^* \| \leq \varepsilon/2\ell_{\Psi}$ and $\min_{\boldsymbol{h} \in \mathcal{D}\varphi(\boldsymbol{\omega}^*)} \| \boldsymbol{h} \| \leq \varepsilon$.

Further, for any arbitrary state distribution $\upsilon \in \Delta(\mathcal{S})$, if $\phi(\upsilon, \cdot)$ is differentiable at $\boldsymbol{\omega}^*$, $\| \nabla_{\boldsymbol{\omega}} \varphi(\upsilon, \boldsymbol{\omega}^*) \| \leq \delta$, i.e., $\boldsymbol{\omega}_{\mathrm{best}}^{(T)}$ is a (ε, δ)-stationary point for the expected exploitability $\phi(\upsilon, \cdot)$.

Chapter 13

Radner Economies

13.1 Background

Having developed a mathematical formalism for Markov pseudo-games, along with a proof of existence of GMPE as well as an algorithm that computes them, we now move on to our main agenda, namely modeling incomplete stochastic economies in this formalism. We establish the first proof, to our knowledge, of the existence of recursive Radner equilibria in Radner economies, and we provide a polynomial-time algorithm for approximating them.

13.1.1 Static Exchange Economies

A **static exchange economy** (or **market**[1]) $(n, m, d, \mathcal{X}, \mathcal{E}, \mathcal{T}, \boldsymbol{u}, \boldsymbol{E}, \boldsymbol{\Theta})$, abbreviated by $(\boldsymbol{E}, \boldsymbol{\Theta})$ when clear from context, comprises a finite set of $n \in \mathbb{N}_+$ **consumers** and $m \in \mathbb{N}_+$ **commodities**. Each consumer $i \in [n]$ arrives at the market with an **endowment** of commodities represented as vector $\boldsymbol{e}_i = (e_{i1}, \dots, e_{im}) \in \mathcal{E}_i$, where $\mathcal{E}_i \subset \mathbb{R}^m$ is called the **endowment space**.[2] Any consumer i can sell its endowment $\boldsymbol{e}_i \in \mathcal{E}_i$ at **prices** $\boldsymbol{p} \in \mathcal{P}$, where $p_j \geq 0$ represents the value (resp. cost) of selling (resp. buying) a unit of commodity $j \in [m]$, to purchase a consumption $\boldsymbol{x}_i \in \mathcal{X}_i$ of commodities in its **consumption space** $\mathcal{X}_i \subseteq \mathbb{R}^m$.[3] Every consumer is

[1]Although a static exchange "market" is an economy, we prefer the term "market" for the static components of a Radner economy, a dynamic exchange economy in which each time-period comprises one static market among many.

[2]Commodities are assumed to include labor services. Further, for any consumer i and endowment $\boldsymbol{e}_i \in \mathcal{E}_i$, $e_{ij} \geq 0$ denotes the quantity of commodity j in consumer i's possession, while $e_{ij} < 0$ denotes consumer i's debt, in terms of commodity j.

[3]We note that, for any labor service j, consumer i's consumption x_{ij} is negative and restricted by its consumption space to be lower bounded by the negative of i's endowment, i.e., $x_{ij} \in [-e_{ij}, 0]$. This modeling choice allows us to model a consumer's preferences over the labor services she can provide. More generally, the consumption space models the constraints imposed on consumption by the "physical properties" of the world. That is, it rules out impossible combinations of commodities, such as strictly

constrained to buy a consumption with a cost weakly less than the value of its endowment, i.e., consumer i's **budget set**—the set of consumptions i can afford with its endowment $\boldsymbol{e}_i \in \mathcal{E}_i$ at prices $\boldsymbol{p} \in \mathcal{P}$—is determined by its **budget correspondence** $\mathcal{B}_i(\boldsymbol{e}_i, \boldsymbol{p}) \doteq \{\boldsymbol{x}_i \in \mathcal{X}_i \mid \boldsymbol{x}_i \cdot \boldsymbol{p} \leq \boldsymbol{e}_i \cdot \boldsymbol{p}\}$.

Each consumer's consumption preferences are determined by its type-dependent preference relation $\succeq_{i,\boldsymbol{\theta}_i}$ on $\mathcal{X}_i$, represented by a type-dependent **utility function** $\boldsymbol{x}_i \mapsto u_i(\boldsymbol{x}_i; \boldsymbol{\theta}_i)$, for **type** $\boldsymbol{\theta}_i \in \mathcal{T}_i$ that characterizes consumer i's preferences within the **type space** $\mathcal{T}_i \subset \mathbb{R}^d$ of possible preferences.[4] The goal of each consumer i is thus to buy a consumption $\boldsymbol{x}_i \in \mathcal{B}_i(\boldsymbol{e}_i, \boldsymbol{p})$ that maximizes its utility function $\boldsymbol{x}_i \mapsto u_i(\boldsymbol{x}_i; \boldsymbol{\theta}_i)$ over its budget set $\mathcal{B}_i(\boldsymbol{e}_i, \boldsymbol{p})$.

We denote any **endowment profile** (resp. **type profile** and **consumption profile**) as $\boldsymbol{E} \doteq (\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n)^T \in \mathcal{E}$ (resp. $\boldsymbol{\Theta} \doteq (\boldsymbol{\theta}_1, \ldots, \boldsymbol{\theta}_n) \in \mathcal{T}$ and $\boldsymbol{X} \doteq (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_n)^T \in \mathcal{X}$). The **aggregate demand** (resp. **aggregate supply**) of a **consumption profile** $\boldsymbol{X} \in \mathcal{X}$ (resp. **an endowment profile** $\boldsymbol{E} \in \mathcal{E}$) is defined as the sum of consumptions (resp. endowments) across all consumers, i.e., $\sum_{i \in [n]} \boldsymbol{x}_i$ (resp. $\sum_{i \in [n]} \boldsymbol{e}_i$).

13.1.2 Radner Economies

A(n infinite horizon) Radner economy (Radner, 1972) $\mathcal{I} \doteq (n, m, l, d, \mathcal{S}, \mathcal{X}, \mathcal{E}, \mathcal{T}, \boldsymbol{u}, \gamma, \rho, \mathcal{R}, \mu)$, comprises $n \in \mathbb{N}$ consumers who, over an infinite discrete time horizon $t = 0, 1, 2, \ldots$, repeatedly encounter the opportunity to buy a consumption of $m \in \mathbb{N}$ commodities and a portfolio of $l \in \mathbb{N}$ assets, with their collective decisions leading them through a **state space** $\mathcal{S} \doteq \mathcal{O} \times (\mathcal{E} \times \mathcal{T})$. This state space comprises a **world state space** $\mathcal{O}$ and a **spot market space** $\mathcal{E} \times \mathcal{T}$. The spot market space is a collection of **spot markets**, each one a static exchange market $(\boldsymbol{E}, \boldsymbol{\Theta}) \in \mathcal{E} \times \mathcal{T} \subseteq \mathbb{R}^m \times \mathbb{R}^d$.

Each **asset** $k \in [l]$ is a **generalized Arrow security**, i.e., a divisible contract that transfers to its owner a quantity of the jth commodity at any world state $o \in \mathcal{O}$ determined by a matrix of asset returns $\boldsymbol{R}_o \doteq (\boldsymbol{r}_{o1}, \ldots, \boldsymbol{r}_{ol})^T \in \mathbb{R}^{l \times m}$ s.t. $r_{okj} \in \mathbb{R}$ denotes the quantity of commodity j transferred at world state o for one unit of asset k. The collection of asset returns across all world states is given by $\mathcal{R} \doteq \{\boldsymbol{R}_o\}_{o \in \mathcal{O}}$. At

positive quantities of a commodity that is not available in the region where a consumer resides, or a supply of labor that amounts to more than 24 labor hours in a given day.

[4] In the sequel, we will be assuming, for any consumer i with any type $\boldsymbol{\theta}_i \in \mathcal{T}_i$, the type-dependent utility function $\boldsymbol{x}_i \mapsto u_i(\boldsymbol{x}_i; \boldsymbol{\theta}_i)$ is continuous, which implies that it can represent any type-dependent preference relation $\succeq_{i,\boldsymbol{\theta}_i}$ on $\mathbb{R}^m$ that is complete, transitive, and continuous (Debreu et al., 1954).

any time step $t = 0, 1, 2, \ldots$, a consumer $i \in [n]$ can invest in an **asset portfolio** $\boldsymbol{y}_i \in \mathcal{Y}_i$ from a **space of asset portfolios (or investments)** $\mathcal{Y}_i \subset \mathbb{R}^l$ that define the **asset market**, where $y_{ik} \geq 0$ denotes the units of asset k bought (long) by consumer i, while $y_{ik} < 0$ denotes units that are sold (short). Assets are assumed to be **short-lived** (Magill and Quinzii, 1994), meaning that any asset purchased at time t pays its dividends in the subsequent time period $t+1$, and then expires.[5] Assets allow consumers to insure themselves against future realizations of the spot market (i.e., types and endowments), by allowing it to transfer wealth across world states.

The economy starts at time period $t = 0$ in an **initial state** $S^{(0)} \sim \mu$ determined by an initial state distribution $\mu \in \Delta(\mathcal{S})$. At each time step $t = 0, 1, 2, \ldots$, the state of the economy is $\boldsymbol{s}^{(t)} \doteq (o^{(t)}, \boldsymbol{E}^{(t)}, \boldsymbol{\Theta}^{(t)}) \in \mathcal{S}$. Each consumer $i \in [n]$, observes the world state $o^{(t)} \in \mathcal{O}$, and participates in a spot market $(\boldsymbol{E}^{(t)}, \boldsymbol{\Theta}^{(t)})$, where it purchases **a consumption** $\boldsymbol{x}_i^{(t)} \in \mathcal{X}_i$ at **commodity prices** $\boldsymbol{p}^{(t)} \in \Delta_m$, and an **asset market** where it invests in an **asset portfolio** $\boldsymbol{y}_i^{(t)} \in \mathcal{Y}_i$ at **assets prices** $\boldsymbol{q}^{(t)} \in \mathbb{R}^l$.[6] Every consumer is constrained to buy a consumption $\boldsymbol{x}_i^{(t)} \in \mathcal{X}_i$ and invest in an asset portfolio $\boldsymbol{y}_i^{(t)} \in \mathcal{Y}_i$ with a total cost weakly less than the value of its current endowment $\boldsymbol{e}_i^{(t)} \in \mathcal{E}_i$. Formally, the set of consumptions and investment portfolios that a consumer i can afford with its current endowment $\boldsymbol{e}_i^{(t)} \in \mathcal{E}_i$ at current commodity prices $\boldsymbol{p}^{(t)} \in \mathcal{P}$ and current asset prices $\boldsymbol{q}^{(t)} \in \mathbb{R}^l$, i.e., its **budget set** $\mathcal{B}_i(\boldsymbol{e}_i^{(t)}, \boldsymbol{p}^{(t)}, \boldsymbol{q}^{(t)})$, is determined by its **budget correspondence**

$$\mathcal{B}_i(\boldsymbol{e}_i, \boldsymbol{p}, \boldsymbol{q}) \doteq \{(\boldsymbol{x}_i, \boldsymbol{y}_i) \in \mathcal{X}_i \times \mathcal{Y}_i \mid \boldsymbol{x}_i \cdot \boldsymbol{p} + \boldsymbol{y}_i \cdot \boldsymbol{q} \leq \boldsymbol{e}_i \cdot \boldsymbol{p}\}.$$

After the consumers make their consumption and investment decisions, they each receive **reward** $u_i(\boldsymbol{x}_i^{(t)}; \boldsymbol{\theta}_i^{(t)})$ as a function of their consumption and type, and then the economy either collapses with probability $1-\gamma$, or survives with probability γ, where $\gamma \in (0,1)$ is called the **discount rate**.[7] If the economy survives to see another day, then a new state is realized, namely $(O', E', \Theta') \sim \rho(\cdot \mid \boldsymbol{s}^{(t)}, \boldsymbol{Y}^{(t)})$,

[5] While for ease of exposition we assume that assets are short-lived, our results generalize to infinitely-lived generalized Arrow securities (Huang and Werner, 2004) (i.e., securities that never expire, so yield returns and can be resold in every subsequent time period following their purchase) with appropriate modifications to the definitions of the budget constraints and Walras' law. In contrast, our results do *not* immediately generalize to k-**period-living generalized Arrow securities** (i.e., securities that yield returns and can be resold in the k subsequent time periods following their purchase, until their expiration), as such securities introduce non-stationarities into the economy. To accommodate such securities would require that we generalize our Markov game model and methods to accommodate policies that depend on histories of length k.

[6] In general, asset prices can be negative. This modeling assumption is in line with the real world: e.g., it is common for energy futures to see negative prices because of costs associated with overproduction and limited storage capacity (Sheppard et al., 2020).

[7] While for ease of exposition we assume a single discount factor for all consumers, our results extend to a setting in which each consumer $i \in [n]$ has a potentially unique discount factor $\gamma_i \in (0,1)$ by incorporating the discount rates into the consumers' payoffs in the Markov pseudo-game defined in Section 14.1.2, rather than the history distribution.

according to a **transition probability function** $\rho : \mathcal{S} \times \mathcal{S} \times \mathcal{Y} \to [0, 1]$ that depends on the consumers' investment portfolio profile $\boldsymbol{Y}^{(t)} \doteq (\boldsymbol{y}_1^{(t)}, \dots, \boldsymbol{y}_n^{(t)})^T \in \mathcal{Y}$, after which the economy transitions to a new state $S^{(t+1)} \doteq (O', E' + \boldsymbol{Y}^{(t)}\boldsymbol{R}_{O'}, \Theta')$, where the consumers' new endowments depends on their returns $\boldsymbol{Y}^{(t)}\boldsymbol{R}_{O'} \in \mathbb{R}^{n \times m}$ on their investments.

Remark 13.1.1.

If only one commodity is delivered in exchange for assets, i.e., for all world states $o \in \mathcal{O}$, $\mathrm{Rank}(\boldsymbol{R}_o) \leq 1$, then the generalized Arrow securities are **numéraire generalized Arrow securities**, and the assets are called **financial assets**.[8] A Radner economy is **world-state-contingent** iff the cardinality of the world state space is weakly greater than that of the spot market space, i.e., $|\mathcal{O}| \geq |\mathcal{E} \times \mathcal{T}|$. Intuitively, when this condition holds, there exists a surjection from world states to spot market states, which implies that spot market states are implicit in world states, so that the spot market states can be dropped from the state space, i.e., $\mathcal{S} \doteq \mathcal{O}$. A Radner economy has **complete asset markets** if it is world-state-contingent, and assets can deliver some commodity at all world states, i.e., for all world states $o \in \mathcal{O}$, $\mathrm{Rank}(\boldsymbol{R}_o) \geq 1$. Otherwise, it has **incomplete asset markets**. Colloquially, we call an infinite horizon exchange economy with (in)complete asset markets an **(in)complete exchange economy**. Intuitively, in complete exchange economies, consumers can insure themselves against all future realizations of the spot market—uncertainty regarding their endowments and types—since a complete exchange economy is world-state contingent. Further, when there is only a single commodity, s.t. $m = 1$, and only one financial asset which is a risk-free bond s.t. $l = 1$, and the return matrix for all world states $o \in \mathcal{O}$ (now a scalar since there is only one commodity and one financial asset) is given by $r_o \doteq \alpha$, for some $\alpha \in \mathbb{R}$, we obtain the standard incomplete market model (Blackwell, 1965; Lucas Jr and Prescott, 1971).

A **history** $\boldsymbol{h} \in \mathcal{H}^\tau \doteq (\mathcal{S} \times \mathcal{X} \times \mathcal{Y} \times \mathcal{P} \times \mathbb{R}^l)^\tau \times \mathcal{S}$ is a sequence $\boldsymbol{h} = ((\boldsymbol{s}^{(t)}, \boldsymbol{X}^{(t)}, \boldsymbol{Y}^{(t)}, \boldsymbol{p}^{(t)}, \boldsymbol{q}^{(t)})_{t=0}^{\tau-1}, \boldsymbol{s}^{(\tau)})$ of tuples comprising states, consumption profiles, investment profiles, commodity price, and asset prices s.t. a history of length 0 corresponds only to the initial state of the economy. For any history $\boldsymbol{h} \in \mathcal{H}^\tau$, we

[8] Recall that the numéraire is a fixed commodity that is used to standardize the value of other commodities, while a numéraire generalized Arrow security is a generalized Arrow security that delivers its returns in terms of the numérarire. If the assets deliver exactly one commodity, i.e., $\mathrm{Rank}(\boldsymbol{R}_o) = 1$ at all world states o, we take that commodity to be the numéraire for the corresponding spot markets. On the other hand, if the assets deliver no commodity, i.e., $\mathrm{Rank}(\boldsymbol{R}_o) = 0$ at world state o, then we can take any arbitrary commodity to be the numéraire, in which case, the assets vacuously "deliver" zero units of the numérarire, and no units of any other commodities either.

denote by $\boldsymbol{h}_{:p}$ the first $p \in [0 : \tau]$ steps of $\boldsymbol{h}$, i.e., $\boldsymbol{h}_{:p} \doteq ((\boldsymbol{s}^{(t)}, \boldsymbol{X}^{(t)}, \boldsymbol{Y}^{(t)}, \boldsymbol{p}^{(t)}, \boldsymbol{q}^{(t)})_{t=0}^{p-1}, \boldsymbol{s}^{(p)})$. Overloading notation, we define the **history space** $\mathcal{H} \doteq \bigcup_{\tau=0}^{\infty} \mathcal{H}^\tau$, and then **consumption**, **investment**, **commodity price** and **asset price policies** as mappings $\boldsymbol{x}_i : \mathcal{H} \to \mathcal{X}_i$, $\boldsymbol{y}_i : \mathcal{H} \to \mathcal{Y}_i$, $\boldsymbol{p} : \mathcal{H} \to \Delta_m$, and $\boldsymbol{q} : \mathcal{H} \to \mathbb{R}^l$ from histories to consumptions, investments, commodity prices, and asset prices, respectively, s.t. $(\boldsymbol{x}_i, \boldsymbol{y}_i)(\boldsymbol{h})$ is the consumption-investment decision of consumer $i \in [n]$, and $(\boldsymbol{p}, \boldsymbol{q})(\boldsymbol{h})$ are commodity and asset prices, both at history $\boldsymbol{h} \in \mathcal{H}$. A **consumption policy profile** (resp. **investment policy profile**) $\boldsymbol{X}(\boldsymbol{h}) \doteq (\boldsymbol{x}_1, \dots, \boldsymbol{x}_n)(\boldsymbol{h})^T$ (resp. $\boldsymbol{Y}(\boldsymbol{h}) \doteq (\boldsymbol{y}_1, \dots, \boldsymbol{y}_n)(\boldsymbol{h})^T$) is a collection of consumption (resp. investment) policies for all consumers. A consumption policy $\boldsymbol{x}_i : \mathcal{S} \to \mathcal{X}_i$ is **Markov** if it depends only on the last state of the history, i.e., $\boldsymbol{x}_i(\boldsymbol{h}) = \boldsymbol{x}_i(\boldsymbol{s}^{(\tau)})$, for all histories $\boldsymbol{h} \in \mathcal{H}^\tau$ of all lengths $\tau \in \mathbb{N}$. An analogous definition extends to investment, commodity price, and asset price policies.

Given $\boldsymbol{\pi} \doteq (\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{p}, \boldsymbol{q})$ and a history $\boldsymbol{h} \in \mathcal{H}^\tau$, we define the **discounted history distribution** assuming initial state distribution μ as

$$\nu_\mu^{\boldsymbol{\pi},\tau}(\boldsymbol{h}) = \mu(\boldsymbol{s}^{(0)}) \prod_{t=0}^{\tau-1} \gamma^t \rho(o^{(t+1)}, \boldsymbol{E}^{(t+1)} + \boldsymbol{Y}^{(t)} \boldsymbol{R}_{o^{(t+1)}}, \boldsymbol{\Theta}^{(t+1)} \mid \boldsymbol{s}^{(t)}, \boldsymbol{Y}^{(t)}) \mathbb{1}_{\{\boldsymbol{Y}(\boldsymbol{h}_{:t})\}}(\boldsymbol{Y}^{(t)}).$$

Overloading notation, we define the set of all realizable trajectories $\mathcal{H}^{\boldsymbol{\pi}}$ of length τ under policy profile $\boldsymbol{\pi}$ as $\mathcal{H}^{\boldsymbol{\pi}} \doteq \operatorname{supp}(\nu_\mu^{\boldsymbol{\pi},\tau})$, i.e., the set of all histories that occur with non-zero probability, and we let $H = \left((S^{(t)}, A^{(t)})_{t=0}^{\tau-1}, S^{(\tau)}\right)$ be any randomly sampled history from $\nu_\mu^{\boldsymbol{\pi},\tau}$. Finally, we abbreviate $\nu_\mu^{\boldsymbol{\pi}} \doteq \nu_\mu^{\boldsymbol{\pi},\infty}$.

13.1.3 Solution Concepts and Existence

An **outcome** $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{p}, \boldsymbol{q}) : \mathcal{H} \to \mathcal{X} \times \mathcal{Y} \times \Delta_m \times \mathbb{R}^l$ of a Radner economy is a tuple consisting of a commodity prices policy, an asset prices policy, a consumption policy profile, and an investment policy profile.[9]

An outcome is **Markov** if all its constituent policies are Markov: i.e., if it depends only on the last state of the history, i.e., $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{p}, \boldsymbol{q})(\boldsymbol{h}) = (\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{p}, \boldsymbol{q})(\boldsymbol{s}^{(\tau)})$, for all histories $\boldsymbol{h} \in \mathcal{H}^\tau$ of all lengths $\tau \in \mathbb{N}$.

We now introduce a number of properties of Radner economies outcomes, which we use to define our solution concepts. While these properties are defined broadly for (in general, history-dependent) outcomes, they also apply in the special case of Markov outcomes.

[9]Instead of expressing this tuple as $\mathcal{X}^{\mathcal{H}} \times \mathcal{Y}^{\mathcal{H}} \times {\Delta_m}^{\mathcal{H}} \times {\mathbb{R}^l}^{\mathcal{H}}$, we sometimes write $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{p}, \boldsymbol{q}) : \mathcal{H} \to \mathcal{X} \times \mathcal{Y} \times \Delta_m \times \mathbb{R}^l$.

Given a consumption and investment profile $(\boldsymbol{X}, \boldsymbol{Y})$, the **consumption state-value function** $v_i^{(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{p},\boldsymbol{q})} : \mathcal{S} \to \mathbb{R}$ is defined as:

$$v_i^{(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{p},\boldsymbol{q})}(\boldsymbol{s}) \doteq \mathop{\mathbb{E}}_{H \sim \nu_\mu^{(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{p},\boldsymbol{q})}} \left[\sum_{t=0}^{\infty} \gamma^t u_i \left(\boldsymbol{x}_i(H_{:t}); \Theta^{(t)} \right) \mid S^{(0)} = \boldsymbol{s} \right].$$

Definition 13.1.1 [Optimal Outcome]**.**

An outcome $(\boldsymbol{X}^*, \boldsymbol{Y}^*, \boldsymbol{p}^*, \boldsymbol{q}^*)$ is **optimal** for i if i's **expected cumulative utility** $u_i(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{p}, \boldsymbol{q}) \doteq \mathbb{E}_{\boldsymbol{s} \sim \mu} \left[v_i^{(\boldsymbol{X},\boldsymbol{Y},\boldsymbol{p},\boldsymbol{q})}(\boldsymbol{s}) \right]$ is maximized over all affordable consumption and investment policies, i.e.,

$$(\boldsymbol{x}_i^*, \boldsymbol{y}_i^*) \in \mathop{\arg\max}_{\substack{(\boldsymbol{x}_i, \boldsymbol{y}_i): \mathcal{H} \to \mathcal{X}_i \times \mathcal{Y}_i, \forall t \in \mathbb{N}, \boldsymbol{h} \in \mathcal{H}^t \\ (\boldsymbol{x}_i, \boldsymbol{y}_i)(\boldsymbol{h}_{:t}) \in \mathcal{B}_i(\boldsymbol{e}_i^{(t)}, \boldsymbol{p}^*(\boldsymbol{h}_{:t}), \boldsymbol{q}^*(\boldsymbol{h}_{:t}))}} u_i(\boldsymbol{x}_i, \boldsymbol{x}_{-i}^*, \boldsymbol{y}_i, \boldsymbol{y}_{-i}^*, \boldsymbol{p}^*, \boldsymbol{q}^*) \ . \tag{13.1}$$

A Markov outcome $(\boldsymbol{X}^*, \boldsymbol{Y}^*, \boldsymbol{p}^*, \boldsymbol{q}^*)$ is **Markov perfect** for i if i maximizes its consumption state-value function over all affordable consumption and investment policies, i.e.,

$$(\boldsymbol{x}_i^*, \boldsymbol{y}_i^*) \in \mathop{\arg\max}_{\substack{(\boldsymbol{x}_i, \boldsymbol{y}_i): \mathcal{S} \to \mathcal{X}_i \times \mathcal{Y}_i : \forall \boldsymbol{s} \in \mathcal{S}, \\ (\boldsymbol{x}_i, \boldsymbol{y}_i)(\boldsymbol{s}) \in \mathcal{B}_i(\boldsymbol{e}_i, \boldsymbol{p}^*(\boldsymbol{s}), \boldsymbol{q}^*(\boldsymbol{s}))}} \left\{ v_i^{(\boldsymbol{x}_i, \boldsymbol{x}_{-i}^*, \boldsymbol{y}_i, \boldsymbol{y}_{-i}^*, \boldsymbol{p}^*, \boldsymbol{q}^*)}(\boldsymbol{s}) \right\} \ . \tag{13.2}$$

Definition 13.1.2 [Feasible Outcomes]**.**

A consumption policy $\boldsymbol{X}$ is said to be **feasible** iff for all time horizons $\tau \in \mathbb{N}$ and histories $\boldsymbol{h} \in \mathcal{H}^\tau$ of length τ,

$$\sum_{i \in [n]} \boldsymbol{x}_i(\boldsymbol{h}) - \sum_{i \in [n]} \boldsymbol{e}_i^{(\tau)} \leq \boldsymbol{0}_m,$$

where $\boldsymbol{e}_i^{(\tau)} \in \mathcal{E}_i$ is consumer i's endowment at the end of history $\boldsymbol{h}$, i.e., at state $\boldsymbol{s}^{(\tau)}$.

Similarly, an investment policy is **feasible** iff for all time horizons $\tau \in \mathbb{N}$ and histories $\boldsymbol{h} \in \mathcal{H}^\tau$ of length τ,

$$\sum_{i \in [n]} \boldsymbol{y}_i(\boldsymbol{h}) \leq \boldsymbol{0}_l.$$

If all the consumption and investment policies associated with an outcome are feasible, we will colloquially refer to the outcome as **feasible** as well.

Definition 13.1.3 [Walras' Law]**.**

An outcome $(\boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{p}, \boldsymbol{q})$ is said to satisfy **Walras' law** iff for all time horizons $\tau \in \mathbb{N}$ and histories $\boldsymbol{h} \in \mathcal{H}^\tau$ of length τ,

$$\boldsymbol{p}(\boldsymbol{h}) \cdot \left(\sum_{i \in [n]} \boldsymbol{x}_i(\boldsymbol{h}) - \sum_{i \in [n]} \boldsymbol{e}_i^{(\tau)} \right) + \boldsymbol{q}(\boldsymbol{h}) \cdot \left(\sum_{i \in [n]} \boldsymbol{y}_i(\boldsymbol{h}) \right) = 0,$$

where, as above, $\boldsymbol{e}_i^{(\tau)} \in \mathcal{E}_i$ is consumer i's endowment at state $\boldsymbol{s}^{(\tau)}$.

The canonical solution concept for stochastic economies is the Radner equilibrium.

Definition 13.1.4 [Radner Equilibrium]**.**

A **Radner (or sequential competitive) equilibrium (RE)** (Radner, 1972) of a Radner economy $\mathcal{I}$ is an outcome $(\boldsymbol{X}^*, \boldsymbol{Y}^*, \boldsymbol{p}^*, \boldsymbol{q}^*)$ that is 1. optimal for all consumers, i.e., Equation (13.1) is satisfied, for all consumers $i \in [n]$; 2. feasible; and 3. satisfies Walras' law.

As a Radner equilibrium is in general infinite dimensional, we are often interested in a recursive Radner equilibrium which is a *Markov* outcome, i.e., one that depends only on the last state of the history rather than the entire history, and as such better behaved.

Definition 13.1.5 [Recursive Radner Equilibrium]**.**

A **recursive Radner (or Walrasian or competitive) equilibrium (RRE)** (Mehra and Prescott, 1977; Prescott and Mehra, 1980) of a Radner economy $\mathcal{I}$ is a Markov outcome $(\boldsymbol{X}^*, \boldsymbol{Y}^*, \boldsymbol{p}^*, \boldsymbol{q}^*)$ that is 1. Markov perfect for all consumers, i.e., Equation (13.2) is satisfied, for all consumers $i \in [n]$; 2. feasible; and 3. satisfies Walras' law.

The following assumptions are standard in the equilibrium literature (see, for instance, Geanakoplos (1990)). We prove the existence of a recursive Radner equilibrium under these assumptions.

Assumption 13.1.1.

Given a Radner economy $\mathcal{I}$, assume for all $i \in [n]$,

1. $\mathcal{X}$, $\mathcal{Y}$, $\mathcal{E}$, are non-empty, closed, convex, with $\mathcal{E}$ additionally bounded;
2. $(\boldsymbol{\theta}_i, \boldsymbol{x}_i) \mapsto u_i(\boldsymbol{x}_i; \boldsymbol{\theta}_i)$ is continuous and concave, and $(\boldsymbol{s}, \boldsymbol{y}_i) \mapsto \rho(\boldsymbol{s}' \mid \boldsymbol{s}, \boldsymbol{y}_i, \boldsymbol{y}_{-i})$ is continuous and stochastically concave, for all $\boldsymbol{s}, \in \mathcal{S}$ and $\boldsymbol{y}_{-i} \in \mathcal{Y}_{-i}$;
3. for all $\boldsymbol{e}_i \in \mathcal{E}_i$, the correspondence

$$(\boldsymbol{p}, \boldsymbol{q}) \rightrightarrows \mathcal{B}_i(\boldsymbol{e}_i, \boldsymbol{p}, \boldsymbol{q}) \cap \{(\boldsymbol{x}_i, \boldsymbol{y}_i) \mid \sum_{i \in [n]} \boldsymbol{x}_i \leq \sum_{i \in [n]} \boldsymbol{e}_i, \sum_{i \in [n]} \boldsymbol{y}_i \leq \boldsymbol{0}_m, (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{X} \times \mathcal{Y}\}$$

is continuous[10];

[10]One way to ensure that this condition holds is to assume that for all $\boldsymbol{s} = (o, \boldsymbol{E}, \boldsymbol{\Theta}) \in \mathcal{S}$, returns from assets are positive $\boldsymbol{R}_o \geq \boldsymbol{0}_{ml}$, and for all consumers $i \in [n]$, there exists $(\boldsymbol{x}_i, \boldsymbol{y}_i) \in \mathcal{X}_i \times \mathcal{Y}_i$, s.t. $\boldsymbol{x}_i < \boldsymbol{e}_i$, $\boldsymbol{y}_i < 0$.

4. $\mathcal{B}_i(\boldsymbol{e}_i, \boldsymbol{p}, \boldsymbol{q}) \cap \{(\boldsymbol{x}_i, \boldsymbol{y}_i) \mid \sum_{i\in[n]} \boldsymbol{x}_i \leq \sum_{i\in[n]} \boldsymbol{e}_i, \sum_{i\in[n]} \boldsymbol{y}_i \leq \boldsymbol{0}_m, (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{X} \times \mathcal{Y}\}$ is non-empty, convex, and compact, for all $\boldsymbol{e}_i \in \mathcal{E}_i$, $\boldsymbol{p} \in \Delta_m$, and $\boldsymbol{q} \in \mathbb{R}^l$[11];
5. (no saturation) there exists an $\boldsymbol{x}_i^+ \in \mathcal{X}_i$ s.t. $u_i(\boldsymbol{x}_i^+; \boldsymbol{\theta}_i) > u_i(\boldsymbol{x}_i; \boldsymbol{\theta}_i)$, for all $\boldsymbol{x}_i \in \mathcal{X}_i$ and $\boldsymbol{\theta}_i \in \mathcal{T}_i$.

Next we associate an **Radner Markov pseudo-game** $\mathcal{M}$ with a given Radner economy $\mathcal{I}$.

Definition 13.1.6 [Radner Markov pseudo-game]**.**

Let $\mathcal{I}$ be a Radner economy. The corresponding **Radner Markov pseudo-game** $\mathcal{M} = (n+1, m+l, \mathcal{S}, \bigtimes_{i\in[n]}(\mathcal{X}_i \times \mathcal{Y}_i) \times (\mathcal{P} \times \mathcal{Q}), \mathcal{B}', r', \rho', \gamma', \mu')$ is defined as

- The $n+1$ players comprise n consumers, players $1, \ldots, n$, and one auctioneer, player $n+1$.
- The set of states $\mathcal{S} = \mathcal{O} \times \mathcal{E} \times \mathcal{T}$. At each state $\boldsymbol{s} = (o, \boldsymbol{E}, \boldsymbol{\Theta}) \in \mathcal{S}$,
  - each consumer $i \in [n]$ chooses an action $\boldsymbol{a}_i = (\boldsymbol{x}_i, \boldsymbol{y}_i) \in \mathcal{B}'_i\left(\boldsymbol{s}, \boldsymbol{a}_{-i}\right) \subseteq \mathcal{X}_i \times \mathcal{Y}_i$ from a set of feasible actions $\mathcal{B}'_i(\boldsymbol{s}, \boldsymbol{a}_{-i}) = \mathcal{B}_i(\boldsymbol{e}_i, \boldsymbol{a}_{n+1}) \cap \{(\boldsymbol{x}_i, \boldsymbol{y}_i) \mid \sum_{i\in[n]} \boldsymbol{x}_i \leq \sum_{i\in[n]} \boldsymbol{e}_i, \sum_{i\in[n]} \boldsymbol{y}_i \leq \boldsymbol{0}_m, (\boldsymbol{X}, \boldsymbol{Y}) \in \mathcal{X} \times \mathcal{Y}\}$ and receives reward $r'_i(\boldsymbol{s}, \boldsymbol{a}) \doteq u_i(\boldsymbol{x}_i; \boldsymbol{\theta}_i)$; and
  - the auctioneer $n+1$ chooses an action $\boldsymbol{a}_{n+1} = (\boldsymbol{p}, \boldsymbol{q}) \in \mathcal{B}'_{n+1}\left(\boldsymbol{s}, \boldsymbol{a}_{-(n+1)}\right) \doteq \mathcal{P} \times \mathcal{Q}$ where $\mathcal{P} \doteq \Delta_m$ and $\mathcal{Q} \subseteq [0, \max_{\boldsymbol{E}\in\mathcal{E}} \sum_{i\in[n]} \sum_{j\in[m]} e_{ij}]^l$, and and receives reward $r'_{n+1}(\boldsymbol{s}, \boldsymbol{a}) \doteq \boldsymbol{p} \cdot \left(\sum_{i\in[n]} \boldsymbol{x}_i - \sum_{i\in[n]} \boldsymbol{e}_i\right) + \boldsymbol{q} \cdot \left(\sum_{i\in[n]} \boldsymbol{y}_i\right)$.
- The transition probability function is defined as $\rho'(\boldsymbol{s}' \mid \boldsymbol{s}, \boldsymbol{a}) \doteq \rho(\boldsymbol{s}' \mid \boldsymbol{s}, \boldsymbol{Y})$.
- The discount rate $\gamma' = \gamma$ and the initial state distribution $\mu' = \mu$.

Our existence proof reformulates the set of recursive Radner equilibria of any Radner economy as the set of GMPE of the Radner Markov pseudo-game.

Theorem 13.1.1.

Consider a Radner economy $\mathcal{I}$. Under Assumption 13.1.1, the set of recursive Radner equilibria of $\mathcal{I}$ is equal to the set of GMPE of the associated Radner Markov pseudo-game $\mathcal{M}$.

Corollary 13.1.1.

Under Assumption 13.1.1, the set of recursive Radner equilibria of a Radner economy is non-empty.

[11]One way to ensure that this condition holds is to assume that for all $\boldsymbol{s} = (o, \boldsymbol{E}, \boldsymbol{\Theta}) \in \mathcal{S}$, returns from assets are positive, i.e., $\boldsymbol{R}_o \geq \boldsymbol{0}_{ml}$, and $\mathcal{X}, \mathcal{Y}$ are bounded from below.

13.1.4 Equilibrium Computation

Since a recursive Radner equilibrium is infinite-dimensional when the state space is continuous, its computation is FNP-hard (Murty and Kabadi, 1987). As such, it is generally believed that the best we can hope to find in polynomial time is an outcome that satisfies the necessary conditions of a stationary point of a recursive Radner equilibrium. Since the set of recursive Radner equilibria of any Radner economy is equal to the set of GMPE of the associated Radner Markov pseudo-game (Theorem 13.1.1), running Algorithm 11 on this Radner Markov pseudo-game will allow us to compute a policy profile that satisfies the necessary conditions of a stationary point of an GMPE, and hence a recursive Radner equilibrium.

Combining Theorem 13.1.1 and Theorem 12.3.1, we thus obtain the following computational complexity guarantees for Algorithm 11, when run on the Radner Markov pseudo-game associated with a Radner economy.[12]

Theorem 13.1.2.

Consider a Radner economy $\mathcal{I}$ and the associated Radner Markov pseudo-games $\mathcal{M}$. Let $(\boldsymbol{\pi}, \boldsymbol{\rho}, \mathbb{R}^{\Omega}, \mathbb{R}^{\Sigma})$ be a parametrization scheme for $\mathcal{M}$ and suppose Assumptions 12.3.2, 12.3.3, and 13.1.1 hold. Then, the convergence results in Theorem 12.3.1 hold for $\mathcal{M}$.

13.2 Experiments

Given a Radner economy $\mathcal{I}$, we associate with it an exchange economy Markov pseudo-game $\mathcal{M}$, and we then construct a neural network to solve $\mathcal{M}$. To do so, we assume a parametrization scheme $(\boldsymbol{\pi}, \boldsymbol{\rho}, \mathbb{R}^{\Omega}, \mathbb{R}^{\Sigma})$, where the parametric policy profiles $(\boldsymbol{\pi}, \boldsymbol{\rho})$ are represented by neural networks with $(\mathbb{R}^{\Omega}, \mathbb{R}^{\Sigma})$ as the corresponding network weights. Computing an RRE via Algorithm 11 can then be seen as the result of training a generative adversarial neural network (Goodfellow et al., 2014), where $\boldsymbol{\pi}$ (resp. $\boldsymbol{\rho}$) is the output of the generator (resp. adversarial) network. We call such a neural representation a **generative adversarial policy network (GAPNet)**.

[12] While for generality and ease of exposition we state Assumptions 12.3.2 and 12.3.3 for the Radner Markov pseudo-game $\mathcal{M}$, we note that when the Radner economy $\mathcal{I}$ satisfies Assumption 13.1.1, to ensure that the associated Radner Markov pseudo-game $\mathcal{M}$ satisfies Assumption 12.3.2 and 12.3.3, it suffices to assume that the parametric policy functions $(\boldsymbol{\pi}, \boldsymbol{\rho})$ are affine; the policy parameter spaces $(\mathbb{R}^{\Omega}, \mathbb{R}^{\Sigma})$ are non-empty, compact, and convex; for all players $i \in [n]$ and types $\boldsymbol{\theta}_i \in \mathcal{T}_i$, the utility function $\boldsymbol{x}_i \mapsto u_i(\boldsymbol{x}_i; \boldsymbol{\theta}_i)$ is twice continuously differentiable; and for all $\boldsymbol{s}, \boldsymbol{s}, \in \mathcal{S}$, the transition function $\boldsymbol{Y} \mapsto \rho(\boldsymbol{s}' \mid \boldsymbol{s}, \boldsymbol{Y})$ is twice continuously differentiable.

Following this approach, we built GAPNets to approximate the RRE in two types of Radner economies: one with a deterministic transition probability function and another with a stochastic transition probability function. Within each type, we experimented with three randomly sampled economies, each with 10 consumers, 10 commodities, 1 asset, 5 world states, and characterized by a distinct class of reward functions, which impart different smoothness properties onto the state-value function: **linear**: $u_i(\boldsymbol{x}_i; \boldsymbol{\theta}_i) = \sum_{j \in [m]} \theta_{ij} x_{ij}$; **Cobb-Douglas**: $u_i(\boldsymbol{x}_i; \boldsymbol{\theta}_i) = \prod_{j \in [m]} {x_{ij}}^{\theta_{ij}}$; and **Leontief**: $u_i(\boldsymbol{x}_i; \boldsymbol{\theta}_i) = \min_{j \in [m]} \left\{ \frac{x_{ij}}{\theta_{ij}} \right\}$.[13]

We compare our results with a classic neural projection method (also known as deep equilibrium nets (Azinovic et al., 2022)), which macroeconomists and others use to solve stochastic economies. In this latter method, one seeks a policy profile that minimizes the norm of the system of first-order necessary and sufficient conditions that characterize RRE (see for instance, (Fernández-Villaverde, 2023)).[14] We use the same network architecture for both methods, and select hyperparameters through grid search. In all experiments, we evaluate the performance of the computed policy profiles using three metrics: total first-order violations, average Bellman errors,[15] and exploitability. For each metric, we randomly sample 50 policy profiles, record their corresponding values, and normalize the results by dividing it by the average.

Figure 13.1 depicts our results for economies with deterministic transition functions. Perhaps unsurprisingly, while GAPNets demonstrates a clear advantage in minimizing exploitability in all three economies, the neural projection method (NPM) slightly outperforms GAPNets in minimizing total first order violations and average Bellman error, the metrics they are specifically designed to minimize. Furthermore, in all three economies, exploitability is near 0, highlighting GAPNet's ability to approximate at least a Radner equilibrium. Figure 13.2 presents our results for economies with stochastic transition functions. These results indicate that stochasticity hinders NPM's ability to minimize the three metrics, even the method is explictly designed to minimize two of them. In contrast, GAPNet successfully minimizes all three metrics across all economies.

[13]Full details of our experimental setup appear in Section 14.2, including hyperparameter search values, final experimental configurations, and visualization code. See also our code repository: `https://github.com/Sadie-Zhao/Markov-Pseudo-Game-EC2025`.

[14]We describe the neural projection method in Section 14.2.1.

[15]The definitions of these two metrics can be found in Section 14.2.1.

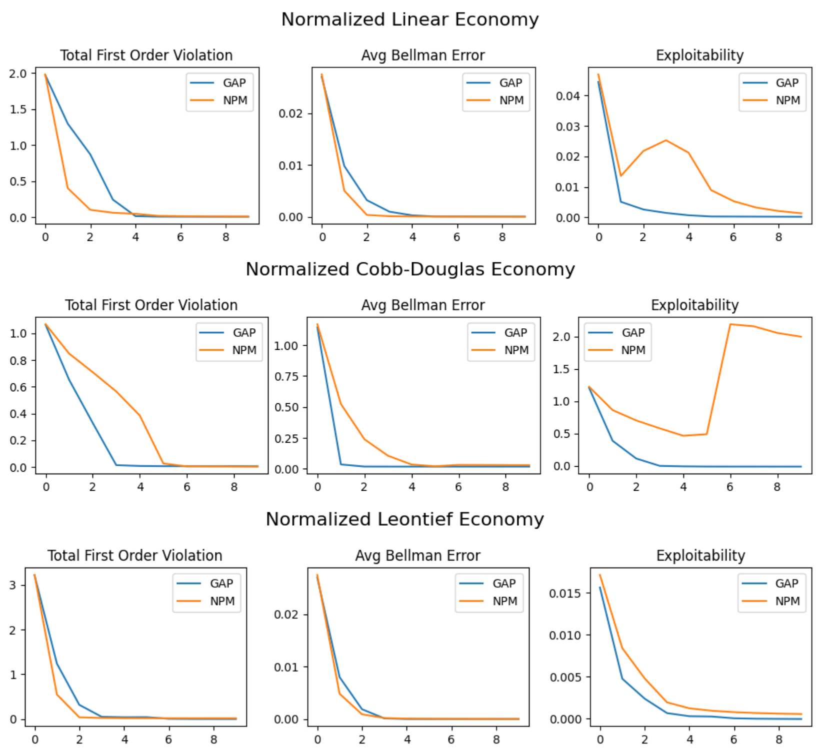


Figure 13.1: Normalized Metrics for Economies with Deterministic Transition Probability Function

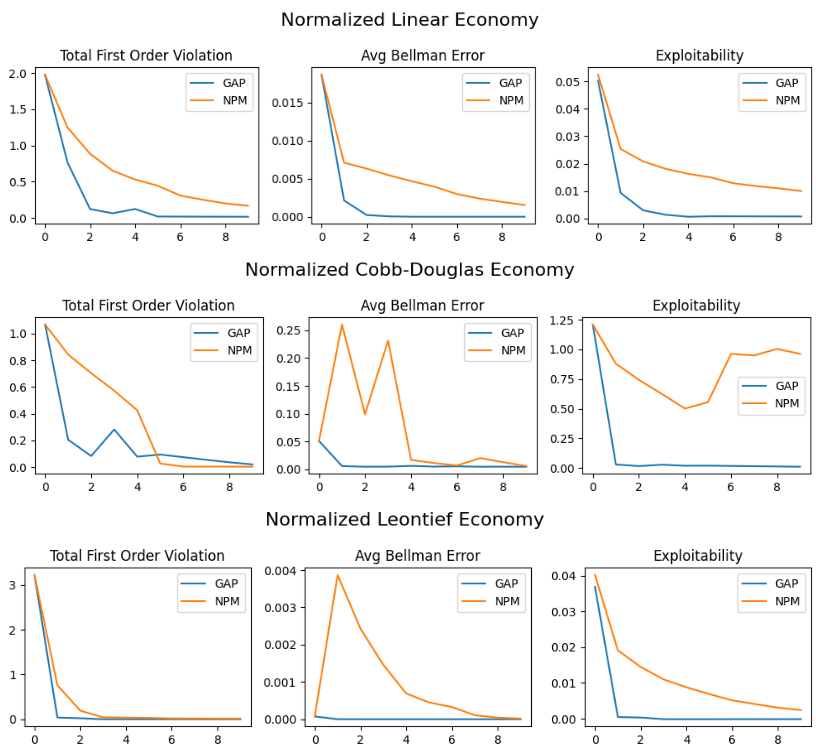


Figure 13.2: Normalized Metrics for Radner Economies with Stochastic Transition Probability Function

Chapter 14

Appendix for Part III

14.1 Omitted Results and Proofs

14.1.1 Omitted Results and Proofs from Chapter 12

Theorem 12.2.1.

Let $\mathcal{M}$ be a Markov pseudo-game for which Assumption 12.2.1 holds, and let $\mathcal{P}^{\text{sub}} \subseteq \mathcal{P}^{\text{markov}}$ be a subspace of Markov policy profiles that satisfies Assumption 12.2.2. Then, there exists a policy $\boldsymbol{\pi}^* \in \mathcal{P}^{\text{sub}}$ such that $\boldsymbol{\pi}^*$ is an GMPE of $\mathcal{M}$.

Proof

First, by Part 3 of Assumption 12.2.1, we know that for any $i \in [n]$, $\mathcal{F}_i^{\text{sub}}(\boldsymbol{\pi}_{-i})$ is non-empty, convex, and compact, for all $\boldsymbol{\pi}_{-i} \in \mathcal{P}_{-i}$. Moreover, 2 of Assumption 12.2.1, $\mathcal{F}^{\text{sub}}$ is upper-hemicontinuous. Therefore, by the Kakutani-Glicksberg fixed-point theorem (see, Theorem 2.4.1—(Glicksberg, 1952)), the set $\mathcal{F}^{\text{sub}} \doteq \{\boldsymbol{\pi} \in \mathcal{P}^{\text{sub}} \mid \boldsymbol{\pi} \in \mathcal{F}^{\text{sub}}(\boldsymbol{\pi})\}$ is non-empty.

For any player $i \in [n]$ and state $\boldsymbol{s} \in \mathcal{S}$, we define the **individual state best-response correspondence** $\Phi_i^{\boldsymbol{s}} : \mathcal{P}^{\text{sub}} \rightrightarrows \mathcal{A}_i$ by

$$\Phi_i^{\boldsymbol{s}}(\boldsymbol{\pi}) \doteq \mathop{\arg\max}_{\boldsymbol{a}_i \in \mathcal{X}_i(\boldsymbol{s}, \boldsymbol{\pi}_{-i}(\boldsymbol{s}))} r_i(\boldsymbol{s}, \boldsymbol{a}_i, \boldsymbol{\pi}_{-i}(\boldsymbol{s})) + \mathop{\mathbb{E}}_{S' \sim \rho(\cdot \mid \boldsymbol{s}, \boldsymbol{a}_i, \boldsymbol{\pi}_{-i}(\boldsymbol{s}))}[\gamma v_i^{\boldsymbol{\pi}}(S')] \tag{14.1}$$

$$= \mathop{\arg\max}_{\boldsymbol{a}_i \in \mathcal{X}_i(\boldsymbol{s}, \boldsymbol{\pi}_{-i}(\boldsymbol{s}))} q_i^{\boldsymbol{\pi}}(\boldsymbol{s}, \boldsymbol{a}_i, \boldsymbol{\pi}_{-i}(\boldsymbol{s})) \tag{14.2}$$

Then, for any player $i \in [n]$, we define the **restricted individual best-response correspondence** $\Phi_i : \mathcal{P}^{\text{sub}} \rightrightarrows \mathcal{P}_i^{\text{sub}}$ as the Cartesian product of individual state best-response correspondences across the states restricted to $\mathcal{P}^{\text{sub}}$:

$$\Phi_i(\boldsymbol{\pi}) = \left(\bigtimes_{\boldsymbol{s} \in \mathcal{S}} \Phi_i^{\boldsymbol{s}}(\boldsymbol{\pi}) \right) \bigcap \mathcal{P}_i^{\text{sub}} \tag{14.3}$$

$$= \{\boldsymbol{\pi}_i \in \mathcal{P}_i^{\text{sub}} \mid \boldsymbol{\pi}_i(\boldsymbol{s}) \in \Phi_i^{\boldsymbol{s}}(\boldsymbol{\pi}), \forall\, \boldsymbol{s} \in \mathcal{S}\} \tag{14.4}$$

Finally, we can define the **multi-player best-response correspondence** $\Phi : \mathcal{P}^{\text{sub}} \rightrightarrows \mathcal{P}^{\text{sub}}$ as the Cartesian product of the individual correspondences, i.e., $\Phi(\boldsymbol{\pi}) \doteq \bigtimes_{i \in [n]} \Phi_i(\boldsymbol{\pi})$.

To show the existence of

MPGNE, we first want to show that there exists a fixed point $\boldsymbol{\pi}^* \in \mathcal{P}^{\text{sub}}$ such that $\boldsymbol{\pi}^* \in \Phi(\boldsymbol{\pi}^*)$. To this end, we need to show that 1. for any $\boldsymbol{\pi} \in \mathcal{P}^{\text{sub}}$, $\Phi(\boldsymbol{\pi})$ is non-empty, compact, and convex; 2. Φ is upper hemicontinuous.

Take any $\boldsymbol{\pi} \in \mathcal{P}^{\text{sub}}$. Fix $i \in [n], \boldsymbol{s} \in \mathcal{S}$, we know that $\boldsymbol{a}_i \mapsto q_i^{\boldsymbol{\pi}}(\boldsymbol{s}, \boldsymbol{a}_i, \boldsymbol{\pi}_{-i}(\boldsymbol{s}))$ is concave over $\mathcal{X}_i(\boldsymbol{s}, \boldsymbol{\pi}_{-i}(\boldsymbol{s}))$, and $\mathcal{X}_i(\boldsymbol{s}, \boldsymbol{\pi}_{-i}(\boldsymbol{s}))$ is non-empty, convex, and compact by Assumption 12.2.1, then by Proposition 4.1 of Fiacco and Kyparisis (1986), $\Phi_i^{\boldsymbol{s}}(\boldsymbol{\pi})$ is non-empty, compact, and convex.

Now, notice $\bigtimes_{\boldsymbol{s} \in \mathcal{S}} \Phi_i^{\boldsymbol{s}}(\boldsymbol{\pi})$ is compact and convex as it is a Cartesian product of compact, convex sets. Thus, as $\mathcal{P}^{\text{sub}}$ is also compact and convex by Assumption 12.2.2, we know that $\Phi_i(\boldsymbol{\pi}) = \left(\bigtimes_{\boldsymbol{s} \in \mathcal{S}} \Phi_i^{\boldsymbol{s}}(\boldsymbol{\pi}) \right) \bigcap \mathcal{P}_i^{\text{sub}}$ is compact and convex. By the assumption of *closure under policy improvement* under Assumption 12.2.2, we know that since $\boldsymbol{\pi} \in \mathcal{P}^{\text{sub}}$, there exists $\boldsymbol{\pi}^+ \in \mathcal{P}^{\text{sub}}$ such that

$$\boldsymbol{\pi}_i^+ \in \operatorname*{arg\,max}_{\boldsymbol{\pi}_i' \in \mathcal{F}_i^{\text{markov}}(\boldsymbol{\pi}_{-i})} q_i^{\boldsymbol{\pi}}(\boldsymbol{s}, \boldsymbol{\pi}_i'(\boldsymbol{s}), \boldsymbol{\pi}_{-i}(\boldsymbol{s}))$$

for all $\boldsymbol{s} \in \mathcal{S}$, and that means $\boldsymbol{\pi}_i^+(\boldsymbol{s}) \in \Phi_i^{\boldsymbol{s}}(\boldsymbol{\pi})$ for all $\boldsymbol{s} \in \mathcal{S}$. Thus, $\Phi_i(\boldsymbol{\pi})$ is also non-empty. Since Cartesian product preserves non-emptiness, compactness, and convexity, we can conclude that $\Phi(\boldsymbol{\pi}) = \bigtimes_{i \in [n]} \Phi_i(\boldsymbol{\pi})$ is non-empty, compact, and convex.

Similarly, fix $i \in [n], \boldsymbol{s} \in \mathcal{S}$, for any $\boldsymbol{\pi} \in \mathcal{P}^{\text{sub}}$, since $\mathcal{X}_i(\boldsymbol{s}, \cdot)$ is continuous (i.e. both upper and lower hemicontinuous), by the Maximum theorem, $\Phi_i^{\boldsymbol{s}}$ is upper hemicontinuous. $\boldsymbol{\pi} \mapsto \bigtimes_{\boldsymbol{s} \in \mathcal{S}} \Phi_i^{\boldsymbol{s}}(\boldsymbol{\pi})$ is upper hemicontinuous as it is a Cartesian product of upper hemicontinuous correspondence, and consequently, $\boldsymbol{\pi} \mapsto \left(\bigtimes_{\boldsymbol{s}\in\mathcal{S}} \Phi_i^{\boldsymbol{s}}(\boldsymbol{\pi})\right) \bigcap \mathcal{P}^{\text{sub}}$ is also upper hemicontinuous. Therefore, Φ is also upper hemicontinuous.

Since $\Phi(\boldsymbol{\pi})$ is non-empty, compact, and convex for any $\boldsymbol{\pi} \in \mathcal{P}^{\text{sub}}$ and Φ is upper hemicontinuous, by the Kakutani-Glicksberg fixed-point theorem (see, Theorem 2.4.1—(Glicksberg, 1952)), Φ admits a fixed point.

Finally, say $\boldsymbol{\pi}^* \in \mathcal{P}^{\text{sub}}$ is a fixed point of Φ, and we want to show that $\boldsymbol{\pi}^*$ is a generalized Markov perfect equilibrium (

MPGNE) of $\mathcal{M}$. Since $\boldsymbol{\pi}^* \in \Phi(\boldsymbol{\pi}^*) = \bigtimes_{i\in[n]} \Phi_i(\boldsymbol{\pi}^*)$, we know that for all $i \in [n]$, $\boldsymbol{\pi}_i^*(\boldsymbol{s}) \in \Phi_i^{\boldsymbol{s}}(\boldsymbol{\pi}^*) = \arg\max_{\boldsymbol{a}_i\in\mathcal{X}_i(\boldsymbol{s},\boldsymbol{\pi}_{-i}^*(\boldsymbol{s}))} q_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}, \boldsymbol{a}_i, \boldsymbol{\pi}_{-i}^*(\boldsymbol{s}))$. We now show that for any $i \in [n]$, for any $\boldsymbol{\pi}_i \in \mathcal{F}_i(\boldsymbol{\pi}_{-i}^*)$, $v_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}) \geq v_i^{(\boldsymbol{\pi}_i,\boldsymbol{\pi}_{-i}^*)}(\boldsymbol{s})$ for all $\boldsymbol{s} \in \mathcal{S}$. Take any policy $\boldsymbol{\pi}_i \in \mathcal{F}_i(\boldsymbol{\pi}_{-i}^*)$. Note that $\boldsymbol{\pi}_i$ may not be Markov, so we denote $\{\boldsymbol{\pi}_i(\boldsymbol{h}_{:t})\}_{t\in\mathbb{N}} = \{\boldsymbol{a}_i^{(t)}\}_{t\in\mathbb{N}}$. Then, for all $\boldsymbol{s}^{(0)} \in \mathcal{S}$,

$$
\begin{aligned}
& v_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}^{(0)}) \\
&= q_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}^{(0)}, \boldsymbol{\pi}_i^*(\boldsymbol{s}^{(0)}), \boldsymbol{\pi}_{-i}^*(\boldsymbol{s}^{(0)})) \\
&= \max_{\boldsymbol{a}_i\in\mathcal{X}_i(\boldsymbol{s}^{(0)},\boldsymbol{\pi}_{-i}^*(\boldsymbol{s}^{(0)}))} q_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}^{(0)}, \boldsymbol{a}_i, \boldsymbol{\pi}_{-i}^*(\boldsymbol{s}^{(0)})) \\
&= \max_{\boldsymbol{a}_i\in\mathcal{X}(\boldsymbol{s}^{(0)},\boldsymbol{\pi}_{-i}^*(\boldsymbol{s}^{(0)}))} r_i(\boldsymbol{s}^{(0)}, \boldsymbol{a}_i, \boldsymbol{\pi}_{-i}^*(\boldsymbol{s}^{(0)})) + \mathop{\mathbb{E}}_{\boldsymbol{s}^{(1)}\sim\rho(\cdot|\boldsymbol{s}^{(0)},\boldsymbol{a}_i,\boldsymbol{\pi}_{-i}^*(\boldsymbol{s}^{(0)}))}[\gamma v_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}^{(1)})] \\
&\geq r_i(\boldsymbol{s}^{(0)}, \boldsymbol{a}_i^{(0)}, \boldsymbol{\pi}_{-i}^*(\boldsymbol{s}^{(0)})) + \mathop{\mathbb{E}}_{\boldsymbol{s}^{(1)}\sim\rho(\cdot|\boldsymbol{s}^{(0)},\boldsymbol{a}_i^{(0)},\boldsymbol{\pi}_{-i}^*(\boldsymbol{s}^{(0)}))}[\gamma v_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}^{(1)})] \qquad (14.5)
\end{aligned}
$$

For any $s^{(0)} \in \mathcal{S}$, define $v_i'(s^{(0)}) \doteq r_i(s^{(0)}, a_i^{(0)}, \pi_{-i}^*(s^{(0)})) + \mathbb{E}_{s^{(1)} \sim \rho(\cdot | s^{(0)}, a_i^{(0)}, \pi_{-i}^*(s^{(0)}))}[\gamma v_i^{\pi^*}(s^{(1)})]$. Since $v_i^{\pi^*}(s) \geq v_i'(s)$ for all $i \in [n]$, $s \in \mathcal{S}$, we have for any $s^{(0)} \in \mathcal{S}$

$$
\begin{aligned}
& v_i^{\pi^*}(s^{(0)}) \\
& \geq r_i(s^{(0)}, a_i^{(0)}, \pi_{-i}^*(s^{(0)})) + \mathop{\mathbb{E}}_{s^{(1)} \sim \rho(\cdot | s^{(0)}, a_i^{(0)}, \pi_{-i}^*(s^{(0)}))} [\gamma v_i^{\pi^*}(s^{(1)})] \\
& \geq r_i(s^{(0)}, a_i^{(0)}, \pi_{-i}^*(s^{(0)})) + \mathop{\mathbb{E}}_{s^{(1)} \sim \rho(\cdot | s^{(0)}, a_i^{(0)}, \pi_{-i}^*(s^{(0)}))} [\gamma v_i'(s^{(1)})] \\
& \geq r_i(s^{(0)}, a_i^{(0)}, \pi_{-i}^*(s^{(0)})) \\
& + \mathop{\mathbb{E}}_{s^{(1)} \sim \rho(\cdot | s^{(0)}, a_i^{(0)}, \pi_{-i}^*(s^{(0)}))} \Bigg[\gamma \Bigg(r_i(s^{(1)}, a_i^{(1)}, \pi_{-i}^*(s^{(1)})) \\
& + \mathop{\mathbb{E}}_{s^{(2)} \sim \rho(\cdot | s^{(1)}, a_i^{(1)}, \pi_{-i}^*(s^{(1)})} [\gamma v_i^{\pi^*}(s^{(2)})] \Bigg) \Bigg] \\
& \geq r_i(s^{(0)}, a_i^{(0)}, \pi_{-i}^*(s^{(0)})) \\
& + \mathop{\mathbb{E}}_{s^{(1)} \sim \rho(\cdot | s^{(0)}, a_i^{(0)}, \pi_{-i}^*(s^{(0)}))} \Bigg[\gamma \Bigg(r_i(s^{(1)}, a_i^{(1)}, \pi_{-i}^*(s^{(1)})) \\
& + \mathop{\mathbb{E}}_{s^{(2)} \sim \rho(\cdot | s^{(1)}, a_i^{(1)}, \pi_{-i}^*(s^{(1)})} [\gamma v_i'(s^{(2)})] \Bigg) \Bigg] \\
& \vdots \\
& \geq v_i^{(\pi_i, \pi_{-i}^*)}(s)
\end{aligned} \tag{14.6}
$$

where in Equation (14.6), we recursively expand v_i' and eliminate v^{π^*} using Equation (14.5). We therefore conclude that for all states $s \in \mathcal{S}$, and for all $i \in [n]$,

$$
v_i^{\pi^*}(s) \geq \max_{\pi_i \in \mathcal{F}_i(\pi_{-i}^*)} v_i^{(\pi_i, \pi_{-i}^*)}(s).
$$

Lemma 12.3.1.

Given a Markov pseudo-game $\mathcal{M}$ for which Assumption 12.2.1 holds, a Markov policy profile $\pi^* \in \mathcal{F}^{\text{markov}}(\pi^*)$ is a GMPE if and only if $\phi(s, \pi^*) = 0$, for all states $s \in \mathcal{S}$. Similarly, a policy profile $\pi^* \in \mathcal{F}(\pi^*)$ is an GNE if and only if $\varphi(\pi^*) = 0$.

Proof of Lemma 12.3.1

We first prove the result for state exploitability.

($\boldsymbol{\pi}^*$ is a

MPGNE $\implies \phi(\boldsymbol{s}, \boldsymbol{\pi}^*) = 0$ for all $\boldsymbol{s} \in \mathcal{S}$): Suppose that $\boldsymbol{\pi}^*$ is a

MPGNE, i.e., for all players $i \in [n]$, $v_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}) \geq \max_{\boldsymbol{\pi}_i \in \mathcal{F}_i(\boldsymbol{\pi}^*_{-i})} v_i^{(\boldsymbol{\pi}_i, \boldsymbol{\pi}^*_{-i})}(\boldsymbol{s})$ for all state $\boldsymbol{s} \in \mathcal{S}$. Then, for all state $\boldsymbol{s} \in \mathcal{S}$, we have

$$\forall i \in [n], \quad \max_{\boldsymbol{\pi}_i \in \mathcal{F}_i(\boldsymbol{\pi}^*_{-i})} v_i^{(\boldsymbol{\pi}_i, \boldsymbol{\pi}^*_{-i})}(\boldsymbol{s}) - v_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}) = 0 \tag{14.7}$$

Summing up across all players $i \in [n]$, we get

$$\phi(\boldsymbol{s}, \boldsymbol{\pi}^*) = \sum_{i \in [n]} \max_{\boldsymbol{\pi}_i \in \mathcal{F}_i(\boldsymbol{\pi}^*_{-i})} v_i^{(\boldsymbol{\pi}_i, \boldsymbol{\pi}^*_{-i})}(\boldsymbol{s}) - v_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}) = 0 \tag{14.8}$$

($\phi(\boldsymbol{s}, \boldsymbol{\pi}^*) = 0$ for all $\boldsymbol{s} \in \mathcal{S} \implies \boldsymbol{\pi}^*$ is a

MPGNE): Suppose we have $\boldsymbol{\pi}^* \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^*)$ and $\phi(\boldsymbol{s}, \boldsymbol{\pi}^*) = 0$ for all $\boldsymbol{s} \in \mathcal{S}$. That is, for any $\boldsymbol{s} \in \mathcal{S}$

$$\phi(\boldsymbol{s}, \boldsymbol{\pi}^*) = \sum_{i \in [n]} \max_{\boldsymbol{\pi}_i \in \mathcal{F}_i(\boldsymbol{\pi}^*_{-i})} v_i^{(\boldsymbol{\pi}_i, \boldsymbol{\pi}^*_{-i})}(\boldsymbol{s}) - v_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}) = 0. \tag{14.9}$$

Since for any $i \in [n]$, $\boldsymbol{\pi}_i^* \in \mathcal{F}_i^{\text{markov}}(\boldsymbol{\pi}^*_{-i})$, $\max_{\boldsymbol{\pi}_i \in \mathcal{F}_i^{\text{markov}}(\boldsymbol{\pi}^*_{-i})} v_i^{(\boldsymbol{\pi}_i, \boldsymbol{\pi}^*_{-i})}(\boldsymbol{s}) - v_i^{\boldsymbol{\pi}^*} \geq v_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}) - v_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}) = 0$. As a result, we must have for all player $i \in [n]$,

$$v_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}) = \max_{\boldsymbol{\pi}_i \in \mathcal{F}(\boldsymbol{\pi}^*_{-i})} v_i^{(\boldsymbol{\pi}_i, \boldsymbol{\pi}^*_{-i})}(\boldsymbol{s}), \quad \forall \boldsymbol{s} \in \mathcal{S} \tag{14.10}$$

Thus, we can conclude that $\boldsymbol{\pi}^*$ is a

MPGNE.

Then, we can prove results for exploitability in an analogous way.

($\boldsymbol{\pi}^*$ is a GNE $\implies \varphi(\boldsymbol{\pi}^*) = 0$): Suppose that $\boldsymbol{\pi}^*$ is a GNE, i.e., for all players $i \in [n]$, $u_i(\boldsymbol{\pi}^*) \geq \max_{\boldsymbol{\pi}_i \in \mathcal{F}_i(\boldsymbol{\pi}^*_{-i})} u_i(\boldsymbol{\pi}_i, \boldsymbol{\pi}^*_{-i})$. Then, we have

$$\forall i \in [n], \quad \max_{\boldsymbol{\pi}_i \in \mathcal{F}_i(\boldsymbol{\pi}^*_{-i})} u_i(\boldsymbol{\pi}_i, \boldsymbol{\pi}^*_{-i}) - u_i(\boldsymbol{\pi}^*) = 0 \tag{14.11}$$

Summing up across all players $i \in [n]$, we get

$$\varphi(\boldsymbol{\pi}^*) = \sum_{i \in [n]} \max_{\boldsymbol{\pi}_i \in \mathcal{F}_i(\boldsymbol{\pi}^*_{-i})} u_i(\boldsymbol{\pi}_i, \boldsymbol{\pi}^*_{-i}) - u_i(\boldsymbol{\pi}^*) = 0 \tag{14.12}$$

($\varphi(\boldsymbol{s}, \boldsymbol{\pi}^*) = 0 \implies \boldsymbol{\pi}^*$ is a GNE): Suppose we have $\boldsymbol{\pi}^* \in \mathcal{F}(\boldsymbol{\pi}^*)$ and $\varphi(\boldsymbol{\pi}^*) = 0$. That is,

$$\varphi(\boldsymbol{\pi}^*) = \sum_{i \in [n]} \max_{\boldsymbol{\pi}_i \in \mathcal{F}_i(\boldsymbol{\pi}^*_{-i})} u_i(\boldsymbol{\pi}_i, \boldsymbol{\pi}^*_{-i}) - u_i(\boldsymbol{\pi}^*) = 0. \tag{14.13}$$

Since for any $i \in [n]$, $\boldsymbol{\pi}^*_i \in \mathcal{F}_i(\boldsymbol{\pi}^*_{-i})$, $\max_{\boldsymbol{\pi}_i \in \mathcal{F}_i(\boldsymbol{\pi}^*_{-i})} u_i(\boldsymbol{\pi}_i, \boldsymbol{\pi}^*_{-i}) - u_i(\boldsymbol{\pi}^*) \geq u_i(\boldsymbol{\pi}^*) - u_i(\boldsymbol{\pi}^*) = 0$. As a result, we must have for all player $i \in [n]$,

$$u_i(\boldsymbol{\pi}^*) = \max_{\boldsymbol{\pi}_i \in \mathcal{F}(\boldsymbol{\pi}^*_{-i})} u_i(\boldsymbol{\pi}_i, \boldsymbol{\pi}^*_{-i}) \tag{14.14}$$

Thus, we can conclude that $\boldsymbol{\pi}^*$ is a GNE.

Observation 12.3.1.

Given a Markov pseudo-game $\mathcal{M}$,

$$\min_{\boldsymbol{\pi} \in \mathcal{F}(\boldsymbol{\pi})} \varphi(\boldsymbol{\pi}) = \min_{\boldsymbol{\pi} \in \mathcal{F}(\boldsymbol{\pi})} \max_{\boldsymbol{\pi}' \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi})} \Psi(\boldsymbol{\pi}, \boldsymbol{\pi}') \ . \tag{12.3}$$

Proof

The per-player maximum operator can be pulled out of the sum in the definition of state-exploitability, because the ith player's best-response policy is independent of the other players' best-response policies, given a fixed policy profile $\boldsymbol{\pi}$:

$$\forall\ \boldsymbol{s} \in \mathcal{S},\ \ \phi(\boldsymbol{s}, \boldsymbol{\pi}) = \sum_{i \in [n]} \max_{\boldsymbol{\pi}'_i \in \mathcal{F}_i^{\text{markov}}(\boldsymbol{\pi}_{-i})} v_i^{(\boldsymbol{\pi}'_i, \boldsymbol{\pi}_{-i})}(\boldsymbol{s}) - v_i^{\boldsymbol{\pi}}(\boldsymbol{s}) \tag{14.15}$$

$$= \max_{\boldsymbol{\pi}' \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi})} \sum_{i \in [n]} v_i^{(\boldsymbol{\pi}'_i, \boldsymbol{\pi}_{-i})}(\boldsymbol{s}) - v_i^{\boldsymbol{\pi}}(\boldsymbol{s}) \tag{14.16}$$

$$= \max_{\boldsymbol{\pi}' \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi})} \psi(\boldsymbol{s}, \boldsymbol{\pi}, \boldsymbol{\pi}') \tag{14.17}$$

The argument is analogous for exploitability:

$$\varphi(\boldsymbol{\pi}) = \sum_{i \in [n]} \max_{\boldsymbol{\pi}'_i \in \mathcal{F}_i^{\text{markov}}(\boldsymbol{\pi}_{-i})} u_i(\boldsymbol{\pi}'_i, \boldsymbol{\pi}_{-i}) - u_i(\boldsymbol{\pi}) \tag{14.18}$$

$$= \max_{\boldsymbol{\pi}' \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi})} \sum_{i \in [n]} u_i(\boldsymbol{\pi}'_i, \boldsymbol{\pi}_{-i}) - u_i(\boldsymbol{\pi}) \tag{14.19}$$

$$= \max_{\boldsymbol{\pi}' \in \mathcal{F}(\boldsymbol{\pi})} \Psi(\boldsymbol{\pi}, \boldsymbol{\pi}') \tag{14.20}$$

Lemma 12.3.3.

Given a Markov pseudo-game $\mathcal{M}$, for $\boldsymbol{\omega} \in \mathbb{R}^\Omega$, suppose that $\phi(\boldsymbol{s}, \cdot)$ is differentiable at $\boldsymbol{\omega}$ for all $\boldsymbol{s} \in \mathcal{S}$. If $\|\nabla_{\boldsymbol{\omega}} \varphi(\boldsymbol{\omega})\| = 0$, then, for all states $\boldsymbol{s} \in \mathcal{S}$, $\|\nabla_{\boldsymbol{\omega}} \phi(\boldsymbol{s}, \boldsymbol{\omega})\| = 0$ μ-almost surely, i.e., $\mu(\{\boldsymbol{s} \in \mathcal{S} \mid \|\nabla_{\boldsymbol{\omega}} \phi(\boldsymbol{s}, \boldsymbol{\omega})\| = 0\}) = 1$. Moreover, for any $\varepsilon > 0$ and $\delta \in [0, 1]$, if $\text{supp}(\mu) = \mathcal{S}$ and $\|\nabla_{\boldsymbol{\omega}} \varphi(\boldsymbol{\omega})\| \leq \varepsilon$, then with probability at least $1 - \delta$, $\|\nabla_{\boldsymbol{\omega}} \phi(\boldsymbol{s}, \boldsymbol{\omega})\| \leq \varepsilon/\delta$.

Proof

First, using Jensen's inequality, by the convexity of the 2-norm $\|\cdot\|$, we have:

$$\begin{aligned} \mathop{\mathbb{E}}_{\boldsymbol{s} \sim \mu} \left[\|\nabla_{\boldsymbol{\omega}} \phi(\boldsymbol{s}, \boldsymbol{\omega})\| \right] &\leq \left\| \mathop{\mathbb{E}}_{\boldsymbol{s} \sim \mu} \left[\nabla_{\boldsymbol{\omega}} \phi(\boldsymbol{s}, \boldsymbol{\omega}) \right] \right\| \\ &= \left\| \nabla_{\boldsymbol{\omega}} \mathop{\mathbb{E}}_{\boldsymbol{s} \sim \mu} \left[\phi(\boldsymbol{s}, \boldsymbol{\omega}) \right] \right\| \\ &= \|\nabla_{\boldsymbol{\omega}} \varphi(\boldsymbol{\omega})\| \ . \end{aligned}$$

The first claim follows directly from the fact that for all $\boldsymbol{s} \in \mathcal{S}$, $\|\nabla_{\boldsymbol{\omega}} \varphi(\boldsymbol{s}, \boldsymbol{\omega})\| \geq 0$, and hence for the expectation $\mathbb{E}_{\boldsymbol{s} \sim \mu} \left[\|\nabla_{\boldsymbol{\omega}} \varphi(\boldsymbol{s}, \boldsymbol{\omega})\| \right]$ to be equal to 0, its value should be equal to zero on a set of measure 1.

Now, for the second part, by Markov's inequality, we have: $\mathbb{P}\left(\|\nabla_{\boldsymbol{\omega}} \phi(\boldsymbol{s}, \boldsymbol{\omega})\| \geq \varepsilon/\delta \right) \leq \frac{\mathbb{E}_{\boldsymbol{s} \sim \mu} \left[\|\nabla_{\boldsymbol{\omega}} \phi(\boldsymbol{s}, \boldsymbol{\pi})\| \right]}{\varepsilon/\delta} \leq \frac{\varepsilon}{\varepsilon/\delta} = \delta$.

Lemma 12.3.4.

Let $\mathcal{M}$ be a Markov pseudo-game with initial state distribution μ. Given policy parameter $\boldsymbol{\omega} \in \mathbb{R}^\Omega$ and arbitrary state distribution $\upsilon \in \Delta(\mathcal{S})$, suppose that both $\phi(\mu, \cdot)$ and $\phi(\upsilon, \cdot)$ are differentiable at $\boldsymbol{\omega}$, then we have: $\|\nabla \phi(\upsilon, \boldsymbol{\omega})\| \leq C_{br}(\boldsymbol{\pi}(\cdot; \boldsymbol{\omega}^*), \mu, \upsilon) \|\nabla \varphi(\boldsymbol{\omega})\|$.

Proof

In this proof, for any $i \in [n]$, we define $\boldsymbol{\sigma}_i(\boldsymbol{\omega}) = \boldsymbol{\rho}_i(\cdot, \boldsymbol{\pi}(\cdot; \boldsymbol{\omega}); \boldsymbol{\sigma})$ as player i's policy in the policy profile $\boldsymbol{\sigma}(\boldsymbol{\omega}) = \boldsymbol{\rho}(\cdot, \boldsymbol{\pi}(\cdot; \boldsymbol{\omega}); \boldsymbol{\sigma})$. Similarly, we define $\boldsymbol{\omega}_i = \boldsymbol{\pi}_i(\cdot; \boldsymbol{\omega})$ as player i's policy in the policy profile $\boldsymbol{\omega} = \boldsymbol{\pi}(\cdot; \boldsymbol{\omega})$.

Given a policy parametrization scheme $(\boldsymbol{\pi}, \boldsymbol{\rho}, \mathbb{R}^\Omega, \mathbb{R}^\Sigma)$, consider any two parameters $\boldsymbol{\omega} \in \mathbb{R}^\Omega, \boldsymbol{\sigma} \in \mathbb{R}^\Sigma$, and any two initial state distributions $\mu, \upsilon \in \Delta(\mathcal{S})$, we know that

$$\left\|\nabla_{\boldsymbol{\omega}} \psi(\upsilon, \boldsymbol{\omega}, \boldsymbol{\sigma})\right\| \tag{14.21}$$

$$= \left\|\nabla_{\boldsymbol{\omega}} \sum_{i \in [n]} u_i(\boldsymbol{\sigma}_i(\boldsymbol{\omega}), \boldsymbol{\omega}_{-i}) - u_i(\boldsymbol{\omega})\right\| \tag{14.22}$$

$$= \left\|\sum_{i \in [n]} \nabla_{\boldsymbol{\omega}} (u_i(\boldsymbol{\sigma}_i(\boldsymbol{\omega}), \boldsymbol{\omega}_{-i}) - u_i(\boldsymbol{\omega}))\right\| \tag{14.23}$$

$$= \left\|\sum_{i \in [n]} \nabla_{\boldsymbol{\omega}} \left[\mathbb{E}_{\substack{\boldsymbol{s}' \sim \delta_\upsilon^{(\boldsymbol{\sigma}_i(\boldsymbol{\omega}), \boldsymbol{\omega}_{-i})} \\ \boldsymbol{s} \sim \delta_\upsilon^{\boldsymbol{\omega}}}} \left[r_i(\boldsymbol{s}', \boldsymbol{\rho}_i(\boldsymbol{s}', \boldsymbol{\pi}(\boldsymbol{s}; \boldsymbol{\omega}); \boldsymbol{\sigma}), \boldsymbol{\pi}_{-i}(\boldsymbol{s}'; \boldsymbol{\omega}) - r_i(\boldsymbol{s}, \boldsymbol{\pi}(\boldsymbol{s}; \boldsymbol{\omega}))\right]\right]\right\| \tag{14.24}$$

$$= \left\|\sum_{i \in [n]} \mathbb{E}_{\substack{\boldsymbol{s}' \sim \delta_\upsilon^{(\boldsymbol{\sigma}_i(\boldsymbol{\omega}), \boldsymbol{\omega}_{-i})} \\ \boldsymbol{s} \sim \delta_\upsilon^{\boldsymbol{\omega}}}} \left[\nabla_{\boldsymbol{a}_{-i}} q_i^{\boldsymbol{\sigma}_i(\boldsymbol{\omega}), \boldsymbol{\omega}_{-i}}(\boldsymbol{s}', \boldsymbol{\rho}_i(\boldsymbol{s}', \boldsymbol{\pi}(\boldsymbol{s}'; \boldsymbol{\omega}); \boldsymbol{\sigma}), \boldsymbol{\pi}_{-i}(\boldsymbol{s}'; \boldsymbol{\omega})) \nabla_{\boldsymbol{\omega}} \left(\boldsymbol{\rho}_i(\boldsymbol{s}', \boldsymbol{\pi}_{-i}(\boldsymbol{s}'; \boldsymbol{\omega}); \boldsymbol{\omega}), \boldsymbol{\pi}(\boldsymbol{s}'; \boldsymbol{\omega})\right)\right.\right.$$

$$\left.\left. - \nabla_{\boldsymbol{a}} q_i^{\boldsymbol{\omega}}(\boldsymbol{s}, \boldsymbol{\pi}(\boldsymbol{s}; \boldsymbol{\omega})) \nabla_{\boldsymbol{\omega}} \boldsymbol{\pi}(\boldsymbol{s}; \boldsymbol{\omega})\right]\right\| \tag{14.25}$$

$$\leq \max_{i \in [n]} \max_{\boldsymbol{s}', \boldsymbol{s} \in \mathcal{S}} \frac{\delta_\upsilon^{(\boldsymbol{\sigma}_i(\boldsymbol{\omega}), \boldsymbol{\omega}_{-i})}(\boldsymbol{s}') \delta_\upsilon^{\boldsymbol{\omega}}(\boldsymbol{s})}{\delta_\mu^{(\boldsymbol{\sigma}_i(\boldsymbol{\omega}), \boldsymbol{\omega}_{-i})}(\boldsymbol{s}') \delta_\mu^{\boldsymbol{\omega}}(\boldsymbol{s})} \left\|\mathbb{E}_{\substack{\boldsymbol{s}' \sim \delta_\mu^{(\boldsymbol{\sigma}_i(\boldsymbol{\omega}), \boldsymbol{\omega}_{-i})} \\ \boldsymbol{s} \sim \delta_\mu^{\boldsymbol{\omega}}}} \left[\nabla_{\boldsymbol{a}_{-i}} q_i^{\boldsymbol{\sigma}_i(\boldsymbol{\omega}), \boldsymbol{\omega}_{-i}}(\boldsymbol{s}', \boldsymbol{\rho}_i(\boldsymbol{s}', \boldsymbol{\pi}(\boldsymbol{s}'; \boldsymbol{\omega}); \boldsymbol{\sigma}), \boldsymbol{\pi}_{-i}(\boldsymbol{s}'; \boldsymbol{\omega}))\right.\right.$$

$$\left.\left. \nabla_{\boldsymbol{\omega}} \left(\boldsymbol{\rho}_i(\boldsymbol{s}', \boldsymbol{\pi}_{-i}(\boldsymbol{s}'; \boldsymbol{\omega}); \boldsymbol{\omega}), \boldsymbol{\pi}(\boldsymbol{s}'; \boldsymbol{\omega})\right) - \nabla_{\boldsymbol{a}} q_i^{\boldsymbol{\omega}}(\boldsymbol{s}, \boldsymbol{\pi}(\boldsymbol{s}; \boldsymbol{\omega})) \nabla_{\boldsymbol{\omega}} \boldsymbol{\pi}(\boldsymbol{s}; \boldsymbol{\omega})\right]\right\| \tag{14.26}$$

$$\leq \max_{i \in [n]} \max_{\boldsymbol{s}', \boldsymbol{s} \in \mathcal{S}} \frac{\delta_\upsilon^{(\boldsymbol{\sigma}_i(\boldsymbol{\omega}), \boldsymbol{\omega}_{-i})}(\boldsymbol{s}') \delta_\upsilon^{\boldsymbol{\omega}}(\boldsymbol{s})}{\delta_\mu^{(\boldsymbol{\sigma}_i(\boldsymbol{\omega}), \boldsymbol{\omega}_{-i})}(\boldsymbol{s}') \delta_\mu^{\boldsymbol{\omega}}(\boldsymbol{s})} \left\|\nabla_{\boldsymbol{\omega}} \left[v_i^{\boldsymbol{\sigma}_i(\boldsymbol{\omega}), \boldsymbol{\omega}_{-i}}(\mu) - v_i^{\boldsymbol{\omega}}(\mu)\right]\right\| \tag{14.27}$$

$$\leq \left(\frac{1}{1-\gamma}\right)^2 \max_{i \in [n]} \max_{\boldsymbol{s}', \boldsymbol{s} \in \mathcal{S}} \frac{\delta_\upsilon^{(\boldsymbol{\sigma}_i(\boldsymbol{\omega}), \boldsymbol{\omega}_{-i})}(\boldsymbol{s}') \delta_\upsilon^{\boldsymbol{\omega}}(\boldsymbol{s})}{\mu(\boldsymbol{s}') \mu(\boldsymbol{s})} \left\|\nabla_{\boldsymbol{\omega}} \psi(\mu, \boldsymbol{\omega}, \boldsymbol{\sigma})\right\| \tag{14.28}$$

$$= \left(\frac{1}{1-\gamma}\right)^2 \max_{i \in [n]} \left\|\frac{\delta_\upsilon^{(\boldsymbol{\sigma}_i(\boldsymbol{\omega}), \boldsymbol{\omega}_{-i})}}{\mu}\right\|_\infty \left\|\frac{\delta_\mu^{\boldsymbol{\omega}}}{\mu}\right\|_\infty \left\|\nabla_{\boldsymbol{\omega}} \psi(\mu, \boldsymbol{\omega}, \boldsymbol{\sigma})\right\| \tag{14.29}$$

where Equation (14.25) and Equation (14.27) are obtained by deterministic policy gradient theorem (Silver et al., 2014), and Equation (14.28) is due to the fact that $\delta_\mu^{\boldsymbol{\omega}}(\boldsymbol{s}) \geq (1-\gamma)\mu(\boldsymbol{s})$ for any $\boldsymbol{\pi} \in \mathcal{P}$, $\boldsymbol{s} \in \mathcal{S}$.

Given condition (1) of Assumption 12.3.3, fix any $\boldsymbol{\omega} \in \mathbb{R}^\Omega$, there exists $\boldsymbol{\sigma}^* \in \mathbb{R}^\Sigma$ s.t. for all $i \in [n]$, $\boldsymbol{s} \in \mathcal{S}$:

$$q_i^{\boldsymbol{\omega}}(\boldsymbol{s}, \boldsymbol{\rho}_i(\boldsymbol{s}, \boldsymbol{\pi}(\boldsymbol{s}; \boldsymbol{\omega}); \boldsymbol{\sigma}^*), \boldsymbol{\pi}_{-i}(\boldsymbol{s}; \boldsymbol{\omega})) = \max_{\boldsymbol{\pi}_i' \in \mathcal{F}_i(\boldsymbol{\pi}(\cdot; \boldsymbol{\omega}))} q_i^{\boldsymbol{\omega}}(\boldsymbol{s}, \boldsymbol{\pi}_i'(\boldsymbol{s}), \boldsymbol{\pi}_{-i}(\boldsymbol{s}; \boldsymbol{\omega})) \ .$$

Thus, $\phi(\boldsymbol{s},\boldsymbol{\omega}) = \psi(\boldsymbol{s},\boldsymbol{\omega},\boldsymbol{\sigma}^*)$ for all $\boldsymbol{s} \in \mathcal{S}$. Hence, plugging in the optimal best-response policy $\boldsymbol{\sigma} = \boldsymbol{\sigma}^*$, we obtain that

$$\|\nabla_{\boldsymbol{\omega}}\phi(\upsilon,\boldsymbol{\omega})\| \leq \left(\frac{1}{1-\gamma}\right)^2 \max_{i\in[n]} \left\|\frac{\delta_\upsilon^{(\boldsymbol{\sigma}_i^*(\boldsymbol{\omega}),\boldsymbol{\omega}_{-i})}}{\mu}\right\|_\infty \left\|\frac{\delta_\mu^{\boldsymbol{\omega}}}{\mu}\right\|_\infty \|\nabla_{\boldsymbol{\omega}}\phi(\mu,\boldsymbol{\omega})\| \tag{14.30}$$

$$\leq \left(\frac{1}{1-\gamma}\right)^2 \max_{i\in[n]} \max_{\boldsymbol{\pi}_i'\in\Phi_i(\boldsymbol{\pi}_{-i}(\cdot;\boldsymbol{\omega}))} \left\|\frac{\delta_\upsilon^{(\boldsymbol{\pi}_i',\boldsymbol{\pi}_{-i}(\cdot;\boldsymbol{\omega}))}}{\mu}\right\|_\infty \left\|\frac{\delta_\mu^{\boldsymbol{\omega}}}{\mu}\right\|_\infty \|\nabla_{\boldsymbol{\omega}}\phi(\mu,\boldsymbol{\omega})\| \tag{14.31}$$

where eq. (14.31) is due to the fact that $\boldsymbol{\sigma}_i^*(\boldsymbol{\omega}) \in \Phi_i(\boldsymbol{\pi}_{-i}(\cdot;\boldsymbol{\omega}))$.

Lemma 12.3.2.

Given a Markov pseudo-game $\mathcal{M}$,

$$\min_{\boldsymbol{\pi}\in\mathcal{F}(\boldsymbol{\pi})} \max_{\boldsymbol{\pi}'\in\mathcal{F}^{\text{markov}}(\boldsymbol{\pi})} \Psi(\boldsymbol{\pi},\boldsymbol{\pi}') = \min_{\boldsymbol{\pi}\in\mathcal{F}(\boldsymbol{\pi})} \max_{\boldsymbol{\rho}\in\mathcal{R}} \Psi(\boldsymbol{\pi},\boldsymbol{\rho}(\cdot,\boldsymbol{\pi}(\cdot))) \ . \tag{12.4}$$

Proof

Fix $\boldsymbol{\pi}^* \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^*)$. We want to show that

$$\max_{\boldsymbol{\pi}'\in\mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^*)} \varphi(\boldsymbol{\pi}^*,\boldsymbol{\pi}') = \max_{\boldsymbol{\rho}\in\mathcal{R}} \varphi(\boldsymbol{\pi}^*,\boldsymbol{\rho}(\cdot,\boldsymbol{\pi}(\cdot))) \ .$$

Define $\mathcal{P}^{\mathcal{R},\boldsymbol{\pi}^*} \doteq \{\boldsymbol{\pi} : \boldsymbol{s} \mapsto \boldsymbol{\rho}(\boldsymbol{s},\boldsymbol{\pi}^*(\boldsymbol{s})) \mid \boldsymbol{\rho} \in \mathcal{R}\} \subseteq \mathcal{P}^{\text{markov}}$.

First, for all $\boldsymbol{\pi}' \in \mathcal{P}^{\mathcal{R},\boldsymbol{\pi}^*}$, $\boldsymbol{\pi}'(\boldsymbol{s}) = \boldsymbol{\rho}(\boldsymbol{s},\boldsymbol{\pi}^*(\boldsymbol{s})) \in \mathcal{X}(\boldsymbol{s},\boldsymbol{\pi}^*(\boldsymbol{s}))$, for all $\boldsymbol{s} \in \mathcal{S}$, by the definition of $\mathcal{R}$. Thus, $\boldsymbol{\pi}' \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^*) = \{\boldsymbol{\pi} \in \mathcal{P}^{\text{markov}} \mid \forall \boldsymbol{s} \in \mathcal{S}, \boldsymbol{\pi}(\boldsymbol{s}) \in \mathcal{X}(\boldsymbol{s},\boldsymbol{\pi}^*(\boldsymbol{s}))\}$. Therefore, $\mathcal{P}^{\mathcal{R},\boldsymbol{\pi}^*} \subseteq \mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^*)$, which implies that $\max_{\boldsymbol{\pi}'\in\mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^*)} \varphi(\boldsymbol{\pi}^*,\boldsymbol{\pi}') \geq \max_{\boldsymbol{\pi}'\in\mathcal{P}^{\mathcal{R},\boldsymbol{\pi}^*}} \varphi(\boldsymbol{\pi}^*,\boldsymbol{\pi}') = \max_{\boldsymbol{\rho}\in\mathcal{R}} \varphi(\boldsymbol{\pi}^*,\boldsymbol{\rho}(\cdot,\boldsymbol{\pi}(\cdot)))$.

Moreover, for all $\boldsymbol{\pi}' \in \mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^*)$, $\boldsymbol{\pi}'(\boldsymbol{s}) \in \mathcal{X}(\boldsymbol{s},\boldsymbol{\pi}^*(\boldsymbol{s}))$, for all $\boldsymbol{s} \in \mathcal{S}$, by the definition of $\mathcal{F}^{\text{markov}}$. Define $\boldsymbol{\rho}'$ such that for all $\boldsymbol{s} \in \mathcal{S}$, $\boldsymbol{\rho}'(\boldsymbol{s},\boldsymbol{a}) = \boldsymbol{\pi}'(\boldsymbol{s})$ if $\boldsymbol{a} = \boldsymbol{\pi}^*(\boldsymbol{s})$, and $\boldsymbol{\rho}'(\boldsymbol{s},\boldsymbol{a}) = \boldsymbol{a}'$ for some $\boldsymbol{a}' \in \mathcal{X}(\boldsymbol{s},\boldsymbol{a})$ otherwise. Note that $\boldsymbol{\rho}' \in \mathcal{R}$, since $\forall(\boldsymbol{s},\boldsymbol{a}) \in \mathcal{S}\times\mathcal{A},\ \boldsymbol{\rho}(\boldsymbol{s},\boldsymbol{a}) \in \mathcal{X}(\boldsymbol{s},\boldsymbol{a})$. Thus, as $\boldsymbol{\pi}'(\boldsymbol{s}) = \boldsymbol{\rho}'(\boldsymbol{s},\boldsymbol{\pi}^*(\boldsymbol{s}))$, for all $\boldsymbol{s} \in \mathcal{S}$, it follows that $\boldsymbol{\pi}' \in \mathcal{P}^{\mathcal{R},\boldsymbol{\pi}^*}$. Therefore, $\mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^*) \subseteq \mathcal{P}^{\mathcal{R},\boldsymbol{\pi}^*}$, which implies that $\max_{\boldsymbol{\pi}'\in\mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^*)} \varphi(\boldsymbol{\pi}^*,\boldsymbol{\pi}') \leq \max_{\boldsymbol{\pi}'\in\mathcal{P}^{\mathcal{R},\boldsymbol{\pi}^*}} \varphi(\boldsymbol{\pi}^*,\boldsymbol{\pi}') = \max_{\boldsymbol{\rho}\in\mathcal{R}} \varphi(\boldsymbol{\pi}^*,\boldsymbol{\rho}(\cdot,\boldsymbol{\pi}(\cdot)))$.

Finally, we conclude that $\max_{\boldsymbol{\pi}'\in\mathcal{F}^{\text{markov}}(\boldsymbol{\pi}^*)} \varphi(\boldsymbol{\pi}^*,\boldsymbol{\pi}') = \max_{\boldsymbol{\rho}\in\mathcal{R}} \varphi(\boldsymbol{\pi}^*,\boldsymbol{\rho}(\cdot,\boldsymbol{\pi}(\cdot)))$.

Theorem 12.3.1.

Given a Markov pseudo-game $\mathcal{M}$, and a parameterization scheme $(\boldsymbol{\pi},\boldsymbol{\rho},\mathbb{R}^\Omega,\mathbb{R}^\Sigma)$, suppose Assumption 12.2.1, 12.3.2, and 12.3.3 hold. For any $\delta > 0$, set $\varepsilon = \delta\|C_{br}(\cdot,\mu,\cdot)\|_\infty^{-1}$. If Algorithm 11 is run with inputs that satisfy, $\eta_{\boldsymbol{\omega}}, \eta_{\boldsymbol{\sigma}} \asymp \text{poly}(\varepsilon, \|\partial\delta_\mu^{\boldsymbol{\pi}^*}/\partial\mu\|_\infty, \frac{1}{1-\gamma}, \ell_{\nabla\Psi}^{-1}, \ell_\Psi^{-1})$, then for some $T \in \text{poly}\left(\varepsilon^{-1}, (1-\gamma)^{-1}, \|\partial\delta_\mu^{\boldsymbol{\pi}^*}/\partial\mu\|_\infty, \ell_{\nabla\Psi}, \ell_\Psi, \text{diam}(\mathbb{R}^\Omega \times \mathbb{R}^\Sigma), \eta_{\boldsymbol{\omega}}^{-1}\right)$, there exists $\boldsymbol{\omega}_{\text{best}}^{(T)} = \boldsymbol{\omega}^{(k)}$ with $k \leq T$ that is a $(\varepsilon, \varepsilon/2\ell_\Psi)$-stationary point of the exploitability, i.e., there exists $\boldsymbol{\omega}^* \in \mathbb{R}^\Omega$ s.t. $\|\boldsymbol{\omega}_{\text{best}}^{(T)} - \boldsymbol{\omega}^*\| \leq \varepsilon/2\ell_\Psi$ and $\min_{\boldsymbol{h}\in\mathcal{D}\varphi(\boldsymbol{\omega}^*)}\|\boldsymbol{h}\| \leq \varepsilon$.

Further, for any arbitrary state distribution $\upsilon \in \Delta(\mathcal{S})$, if $\phi(\upsilon, \cdot)$ is differentiable at $\boldsymbol{\omega}^*$, $\|\nabla_{\boldsymbol{\omega}}\varphi(\upsilon, \boldsymbol{\omega}^*)\| \leq \delta$, i.e., $\boldsymbol{\omega}_{\text{best}}^{(T)}$ is a (ε, δ)-stationary point for the expected exploitability $\phi(\upsilon, \cdot)$.

Proof

As is common in the optimization literature (see, for instance, Davis et al. (2018)), we consider the Moreau envelope of the exploitability, which we simply call the **Moreau exploitability**, i.e.,

$$\tilde{\varphi}(\boldsymbol{\omega}) \doteq \min_{\boldsymbol{\omega}'\in\mathbb{R}^\Omega} \left\{\varphi(\boldsymbol{\omega}') + \ell_{\nabla\psi}\left\|\boldsymbol{\omega} - \boldsymbol{\omega}'\right\|^2\right\} \quad .$$

Similarly, we also consider the **state Moreau exploitability**, i.e., the Moreau envelope of the state exploitability:

$$\tilde{\phi}(\boldsymbol{s}, \boldsymbol{\omega}) \doteq \min_{\boldsymbol{\omega}'\in\mathbb{R}^\Omega} \left\{\phi(\boldsymbol{s}, \boldsymbol{\omega}') + \ell_{\nabla\psi}\left\|\boldsymbol{\omega} - \boldsymbol{\omega}'\right\|^2\right\} \quad .$$

We recall that in these definitions, by our notational convention, $\ell_{\nabla\psi} \geq 0$, refers to the Lipschitz-smoothness constants of the state exploitability which in this case we take to be the largest across all states, i.e., for all $\boldsymbol{s} \in \mathcal{S}$, $(\boldsymbol{\omega}, \boldsymbol{\sigma}) \mapsto \psi(\boldsymbol{s}, \boldsymbol{\omega}, \boldsymbol{\sigma})$ is $\ell_{\nabla\psi}$-Lipschitz-smooth, respectively, and which we note is guaranteed to exist under Assumption 12.3.2. Further, we note that since $\Psi(\boldsymbol{\omega}, \boldsymbol{\sigma}) = \mathbb{E}_{\boldsymbol{s}\sim\mu}\left[\psi(\boldsymbol{s}, \boldsymbol{\omega}, \boldsymbol{\sigma})\right]$ is a weighted average of ψ, $(\boldsymbol{\omega}, \boldsymbol{\sigma}) \mapsto \Psi(\boldsymbol{\omega}, \boldsymbol{\sigma})$ is also $\ell_{\nabla\psi}$-Lipschitz-smooth.

We invoke Theorem 2 of Daskalakis et al. (2020a). Although their result is stated for gradient-dominated-gradient-dominated functions, their proof applies in the more general case of non-convex-gradient-dominated functions.

First, Assumption 12.3.2 guarantees that the cumulative regret Ψ is Lipschitz-smooth w.r.t. $(\boldsymbol{\omega}, \boldsymbol{\sigma})$. Moreover, under Assumption 12.3.2, which guarantees that $\boldsymbol{\sigma} \mapsto q_i^{\boldsymbol{\omega}'}(\boldsymbol{s}, \boldsymbol{\rho}_i(\boldsymbol{s}, \boldsymbol{\pi}_{-i}(\boldsymbol{s}; \boldsymbol{\omega}); \boldsymbol{\sigma}), \boldsymbol{\pi}_{-i}(\boldsymbol{s}; \boldsymbol{\omega}))$ is continuously differentiable for all $\boldsymbol{s} \in \mathcal{S}$ and $\boldsymbol{\omega}, \boldsymbol{\omega}' \in \mathbb{R}^\Omega$, and Assumption 12.3.3, we have that Ψ is $\left(\left\|\partial\delta_\mu^{\boldsymbol{\pi}^*}/\partial\mu\right\|_\infty/1-\gamma\right)$-gradient-dominated in $\boldsymbol{\sigma}$, for all $\boldsymbol{\omega} \in \mathbb{R}^\Omega$, by Theorems 2 and 4 of Bhandari and Russo (2019). Finally, under Assumption 12.3.2, since the policy, the reward function, and the transition probability function are all Lipschitz-continuous, $\widehat{u}$, $\widehat{\Psi}$, and hence $\widehat{G}$ are also Lipschitz-continuous, since $\mathcal{S}$ and $\mathcal{A}$ are compact. Their variance must therefore be bounded, i.e., there exists $\varsigma_{\boldsymbol{\omega}}, \varsigma_{\boldsymbol{\sigma}} \in \mathbb{R}$ s.t. $\mathbb{E}_{\boldsymbol{h},\boldsymbol{h}'}[\widehat{G_{\boldsymbol{\omega}}}(\boldsymbol{\omega}, \boldsymbol{\sigma}; \boldsymbol{h}, \boldsymbol{h}') - \nabla_{\boldsymbol{\omega}} \Psi(\boldsymbol{\omega}, \boldsymbol{\sigma}; \boldsymbol{h}, \boldsymbol{h}')] \leq \varsigma_{\boldsymbol{\omega}}$ and $\mathbb{E}_{\boldsymbol{h},\boldsymbol{h}'}[\widehat{G_{\boldsymbol{\sigma}}}(\boldsymbol{\omega}, \boldsymbol{\sigma}; \boldsymbol{h}, \boldsymbol{h}') - \nabla_{\boldsymbol{\sigma}} \Psi(\boldsymbol{\omega}, \boldsymbol{\sigma}; \boldsymbol{h}, \boldsymbol{h}')] \leq \varsigma_{\boldsymbol{\sigma}}$. Hence, under our assumptions, the assumptions of Theorem 2 of Daskalakis et al. are satisfied. Therefore, $1/T+1 \sum_{t=0}^{T} \|\nabla \widetilde{\varphi}(\boldsymbol{\omega}^{(t)})\| \leq \varepsilon$. Taking a minimum across all $t \in [T]$, we conclude $\left\|\nabla \widetilde{\varphi}(\boldsymbol{\omega}_{\text{best}}^{(T)})\right\| \leq \varepsilon$. Then, by the Lemma 3.7 of (Lin et al., 2020), there exists some $\boldsymbol{\omega}^* \in \mathbb{R}^{\Omega}$ such that $\|\boldsymbol{\omega}_{\text{best}}^{(T)} - \boldsymbol{\omega}^*\| \leq \frac{\varepsilon}{2\ell_\Psi}$ and $\boldsymbol{\omega}^* \in \mathbb{R}^{\Omega}_{\varepsilon} \doteq \{\boldsymbol{\omega} \in \mathbb{R}^{\Omega} \mid \exists \alpha \in \mathcal{D}\varphi(\boldsymbol{\omega}), \|\alpha\| \leq \varepsilon\}$. That is, $\boldsymbol{\omega}_{\text{best}}^{(T)}$ is a $(\varepsilon, \frac{\varepsilon}{2\ell_\Psi})$-stationary point of φ. Furthermore, if we assume that $\phi(\delta, \cdot)$ is differentiable at $\boldsymbol{\omega}^*$ for any state distribution $\delta \in \Delta(\mathcal{S})$, φ is also differentiable at $\boldsymbol{\omega}^*$. Hence, by the proof of Lemma 12.3.4, we know that for any state distribution $\upsilon \in \Delta(\mathcal{S})$,

$$\|\nabla_{\boldsymbol{\omega}} \phi(\upsilon, \boldsymbol{\omega})\| \leq \max_{\boldsymbol{\sigma}^* \in \arg\max_{\boldsymbol{\sigma} \in \mathbb{R}^{\Sigma}} \psi(\upsilon, \boldsymbol{\omega}, \boldsymbol{\sigma})} \|\nabla_{\boldsymbol{\omega}} \psi(\upsilon, \boldsymbol{\omega}, \boldsymbol{\sigma}^*)\| \tag{14.32}$$

$$\leq \max_{i \in [n]} \max_{\boldsymbol{\sigma}^* \in \arg\max_{\boldsymbol{\sigma} \in \mathbb{R}^{\Sigma}} \psi(\upsilon, \boldsymbol{\omega}, \boldsymbol{\sigma})} \tag{14.33}$$

$$\left(\frac{1}{1-\gamma}\right)^2 \left\|\frac{\delta_{\upsilon}^{\boldsymbol{\sigma}_i^*(\boldsymbol{\omega}), \boldsymbol{\omega}_{-i}}}{\mu}\right\|_{\infty} \left\|\frac{\delta_{\upsilon}^{\boldsymbol{\omega}}}{\mu}\right\|_{\infty} \|\nabla_{\boldsymbol{\omega}} \Psi(\boldsymbol{\omega}, \boldsymbol{\sigma}^*)\| \tag{14.34}$$

$$= C_{br}(\boldsymbol{\omega}, \mu, \upsilon) \|\nabla_{\boldsymbol{\omega}} \Psi(\boldsymbol{\omega}, \boldsymbol{\sigma}^*)\| \tag{14.35}$$

$$\frac{1}{C_{br}(\boldsymbol{\omega}, \mu, \upsilon)} \|\nabla_{\boldsymbol{\omega}} \phi(\upsilon, \boldsymbol{\omega})\| \leq \|\nabla_{\boldsymbol{\omega}} \Psi(\boldsymbol{\omega}, \boldsymbol{\sigma}^*)\| \tag{14.36}$$

Therefore,

$$\boldsymbol{\omega}^* \in \mathbb{R}^{\Omega}_{\varepsilon} \doteq \{\boldsymbol{\omega} \in \mathbb{R}^{\Omega} \mid \exists \alpha \in \mathcal{D}\varphi(\boldsymbol{\omega}), \|\alpha\| \leq \varepsilon\} \tag{14.37}$$

$$\supseteq \{\boldsymbol{\omega} \in \mathbb{R}^{\Omega} \mid \exists \boldsymbol{\sigma}^* \in \arg\max_{\boldsymbol{\omega} \in \mathbb{R}^{\Omega}} \Psi(\boldsymbol{\omega}, \boldsymbol{\sigma}) s.t. \|\nabla_{\boldsymbol{\omega}} \Psi(\boldsymbol{\omega}, \boldsymbol{\sigma}^*)\| \leq \varepsilon\} \tag{14.38}$$

$$\supseteq \{\boldsymbol{\omega} \in \mathbb{R}^{\Omega} \mid 1/C_{br}(\boldsymbol{\omega}, \mu, \upsilon) \|\nabla_{\boldsymbol{\omega}} \phi(\upsilon, \boldsymbol{\omega})\| \leq \varepsilon\} \tag{14.39}$$

$$= \{\boldsymbol{\omega} \in \mathbb{R}^{\Omega} \mid \|\nabla_{\boldsymbol{\omega}} \phi(\upsilon, \boldsymbol{\omega})\| \leq \delta\} \tag{14.40}$$

Therefore, we can conclude that there exists $\boldsymbol{\omega}^*$ such that $\|\boldsymbol{\omega}_{\text{best}}^{(T)} - \boldsymbol{\omega}^*\| \leq \frac{\varepsilon}{2\ell_\Psi}$ and $\|\nabla_{\boldsymbol{\omega}} \phi(\upsilon, \boldsymbol{\omega})\| \leq \delta$ for any υ. Thus, $\boldsymbol{\omega}_{\text{best}}^{(T)}$ is a (ε, δ)-stationary point of $\phi(\upsilon, \cdot)$ for any $\upsilon \in \Delta(\mathcal{S})$.

14.1.2 Omitted Results and Proofs from Section 13.1.3

Theorem 13.1.1.

Consider a Radner economy $\mathcal{I}$. Under Assumption 13.1.1, the set of recursive Radner equilibria of $\mathcal{I}$ is equal to the set of GMPE of the associated Radner Markov pseudo-game $\mathcal{M}$.

Proof

Let $\boldsymbol{\pi}^* = (\boldsymbol{X}^*, \boldsymbol{Y}^*, \boldsymbol{p}^*, \boldsymbol{q}^*) : \mathcal{S} \to \mathcal{X} \times \mathcal{Y} \times \mathcal{P} \times \mathcal{Q}$ be an GMPE of the Radner Markov pseudo-game $\mathcal{M}$ associated with $\mathcal{I}$. We want to show that it is also an RRE of $\mathcal{I}$.

First, we want to show that $\boldsymbol{\pi}^*$ is Markov perfect for all consumers. We can make some easy observations: the state value for the player $i \in [n]$ in the Radner Markov pseudo-game at state $\boldsymbol{s} \in \mathcal{S}$ induced by the policy $\boldsymbol{\pi}^*$

$$v_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}) = \mathop{\mathbb{E}}_{H \sim \nu^{\boldsymbol{\pi}^*}} \left[\sum_{t=0}^{\infty} \gamma^t r'(S^{(t)}, A^{(t)}) \mid S^{(0)} = \boldsymbol{s} \right] \tag{14.41}$$

$$= \mathop{\mathbb{E}}_{H \sim \nu^{\boldsymbol{\pi}^*}} \left[\sum_{t=0}^{\infty} \gamma^t u_i(\boldsymbol{x}_i^*(S^{(t)}); \Theta_i^{(t)}) \mid S^{(0)} = \boldsymbol{s} \right] \tag{14.42}$$

is equal to the consumption state value induced by $(\boldsymbol{X}^*, \boldsymbol{Y}^*, \boldsymbol{p}^*, \boldsymbol{q}^*)$

$$v_i^{(\boldsymbol{X}^*, \boldsymbol{Y}^*, \boldsymbol{p}^*, \boldsymbol{q}^*)}(\boldsymbol{s}) \doteq \mathop{\mathbb{E}}_{H \sim \nu^{(\boldsymbol{X}^*, \boldsymbol{Y}^*, \boldsymbol{p}^*, \boldsymbol{q}^*)}} \left[\sum_{t=0}^{\infty} \gamma^t u_i\left(\boldsymbol{x}_i^*(H_{:t}); \Theta^{(t)}\right) \mid S^{(0)} = \boldsymbol{s} \right] . \tag{14.43}$$

as $\boldsymbol{x}_i^*$ is Markov. Since $\boldsymbol{\pi}^*$ is a GMPE, we know that for any $i \in [n]$:

$$(\boldsymbol{x}_i^*, \boldsymbol{y}_i^*) \in \mathop{\arg\max}_{\substack{(\boldsymbol{x}_i, \boldsymbol{y}_i) : \mathcal{S} \to \mathcal{X}_i \times \mathcal{Y}_i : \forall \boldsymbol{s} \in \mathcal{S}, \\ (\boldsymbol{x}_i, \boldsymbol{y}_i)(\boldsymbol{s}) \in \mathcal{B}_i(\boldsymbol{e}_i, \boldsymbol{p}^*(\boldsymbol{s}), \boldsymbol{q}^*(\boldsymbol{s}))}} \left\{ v_i^{(\boldsymbol{x}_i, \boldsymbol{x}_{-i}^*, \boldsymbol{y}_i, \boldsymbol{y}_{-i}^*, \boldsymbol{p}^*, \boldsymbol{q}^*)}(\boldsymbol{s}) \right\}$$

for all $\boldsymbol{s} \in \mathcal{S}$, so $(\boldsymbol{X}^*, \boldsymbol{Y}^*, \boldsymbol{p}^*, \boldsymbol{q}^*)$ is Markov perfect.

Next, we want to show that $(\boldsymbol{X}^*, \boldsymbol{Y}^*, \boldsymbol{p}^*, \boldsymbol{q}^*)$ satisfies the Walras's law. First, we show that for any $i \in [n]$, $\boldsymbol{s} \in \mathcal{S}$, $\boldsymbol{x}_i^*(\boldsymbol{s}) \cdot \boldsymbol{p}^*(\boldsymbol{s}) + \boldsymbol{y}_i^*(\boldsymbol{s}) \cdot \boldsymbol{q}^*(\boldsymbol{s}) - \boldsymbol{e}_i \cdot \boldsymbol{p}^*(\boldsymbol{s}) = 0$. By way of contradiction, assume that there exists some $i \in [n]$, $\boldsymbol{s} \in \mathcal{S}$ such that $\boldsymbol{x}_i^*(\boldsymbol{s}) \cdot \boldsymbol{p}^*(\boldsymbol{s}) + \boldsymbol{y}_i^*(\boldsymbol{s}) \cdot \boldsymbol{q}^*(\boldsymbol{s}) - \boldsymbol{e}_i \cdot \boldsymbol{p}^*(\boldsymbol{s}) \neq 0$. Note that $(\boldsymbol{x}_i^*(\boldsymbol{s}), \boldsymbol{y}_i^*(\boldsymbol{s})) \in \mathcal{B}'(\boldsymbol{s}, \boldsymbol{a}_{-i}) = \mathcal{B}(\boldsymbol{e}_i, \boldsymbol{p}^*(\boldsymbol{s}), \boldsymbol{q}^*(\boldsymbol{s})) = \{(\boldsymbol{x}_i, \boldsymbol{y}_i) \in \mathcal{X}_i \times \mathcal{Y}_i \mid \boldsymbol{x}_i \cdot \boldsymbol{p}^*(\boldsymbol{s}) + \boldsymbol{y}_i \cdot \boldsymbol{q}^*(\boldsymbol{s}) \leq \boldsymbol{e}_i \cdot \boldsymbol{p}^*(\boldsymbol{s})\}$, so we must have $\boldsymbol{x}_i^*(\boldsymbol{s}) \cdot \boldsymbol{p}^*(\boldsymbol{s}) + \boldsymbol{y}_i^*(\boldsymbol{s}) \cdot \boldsymbol{q}^*(\boldsymbol{s}) - \boldsymbol{e}_i \cdot \boldsymbol{p}^*(\boldsymbol{s}) < 0$. By the (no saturation) condition of Assumption 13.1.1, there exists $\boldsymbol{x}_i^+ \in \mathcal{X}_i$ s.t. $u_i(\boldsymbol{x}_i^+; \boldsymbol{\theta}_i) > u_i(\boldsymbol{x}_i^*(\boldsymbol{s}); \boldsymbol{\theta}_i)$. Moreover, since $\boldsymbol{x}_i \mapsto u_i(\boldsymbol{x}_i; \boldsymbol{\theta}_i)$ is concave, for any $0 < t < 1$, $u_i(t\boldsymbol{x}_i^+ + (1-t)\boldsymbol{x}_i^*(\boldsymbol{s}); \boldsymbol{\theta}_i) > u_i(\boldsymbol{x}_i^*(\boldsymbol{s}); \boldsymbol{\theta}_i)$. Since

$\boldsymbol{x}_i^*(\boldsymbol{s})\cdot\boldsymbol{p}^*(\boldsymbol{s})+\boldsymbol{y}_i^*(\boldsymbol{s})\cdot\boldsymbol{q}^*(\boldsymbol{s})-\boldsymbol{e}_i\cdot\boldsymbol{p}^*(\boldsymbol{s}) < 0$, we can pick t small enough such that $\boldsymbol{x}_i' = t\boldsymbol{x}_i^+ + (1-t)\boldsymbol{x}_i^*(\boldsymbol{s})$ satisfies $\boldsymbol{x}_i^{'}(\boldsymbol{s})\cdot\boldsymbol{p}^*(\boldsymbol{s})+\boldsymbol{y}_i^*(\boldsymbol{s})\cdot\boldsymbol{q}^*(\boldsymbol{s})-\boldsymbol{e}_i\cdot\boldsymbol{p}^*(\boldsymbol{s}) \leq 0$ but $\boldsymbol{x}_i' \in \mathcal{X}_i$ s.t. $u_i(\boldsymbol{x}_i^+;\boldsymbol{\theta}_i) > u_i(\boldsymbol{x}_i^*(\boldsymbol{s});\boldsymbol{\theta}_i)$. Thus,

$$
\begin{aligned}
& q_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}, \boldsymbol{x}_i', \boldsymbol{x}_{-i}^*(\boldsymbol{s}), \boldsymbol{Y}^*(\boldsymbol{s}), \boldsymbol{p}^*(\boldsymbol{s}), \boldsymbol{q}^*(\boldsymbol{s})) && (14.44)\\
&= r_i'(\boldsymbol{s}, \boldsymbol{x}_i', \boldsymbol{x}_{-i}^*(\boldsymbol{s}), \boldsymbol{Y}^*(\boldsymbol{s}), \boldsymbol{p}^*(\boldsymbol{s}), \boldsymbol{q}^*(\boldsymbol{s})) + \mathop{\mathbb{E}}_{S'\sim\rho(S'|\boldsymbol{s},\boldsymbol{Y}^*(\boldsymbol{s}))}[\gamma v_i^{\boldsymbol{\pi}^*}(S')] && (14.45)\\
&= u_i(\boldsymbol{x}_i';\boldsymbol{\theta}_i) + \mathop{\mathbb{E}}_{S'\sim\rho(S'|\boldsymbol{s},\boldsymbol{Y}^*(\boldsymbol{s}))}[\gamma v_i^{\boldsymbol{\pi}^*}(S')] && (14.46)\\
&> u_i(\boldsymbol{x}_i^*(\boldsymbol{s});\boldsymbol{\theta}_i) + \mathop{\mathbb{E}}_{S'\sim\rho(S'|\boldsymbol{s},\boldsymbol{Y}^*(\boldsymbol{s}))}[\gamma v_i^{\boldsymbol{\pi}^*}(S')] && (14.47)\\
&= q_i^{\boldsymbol{\pi}^*}(\boldsymbol{s}, \boldsymbol{X}^*(\boldsymbol{s}), \boldsymbol{Y}^*(\boldsymbol{s}), \boldsymbol{p}^*(\boldsymbol{s}), \boldsymbol{q}^*(\boldsymbol{s})) && (14.48)
\end{aligned}
$$

This contradicts that fact that $\boldsymbol{\pi}^*$ is a GMPE since an optimal policy is supposed to be greedy optimal (i.e., maximize the action-value function of each player over its action space at all states) respect to optimal action value function. Thus, we know that for all $i \in [n]$, $\boldsymbol{s} \in \mathcal{S}$, $\boldsymbol{x}_i^*(\boldsymbol{s})\cdot\boldsymbol{p}^*(\boldsymbol{s})+\boldsymbol{y}_i^*(\boldsymbol{s})\cdot\boldsymbol{q}^*(\boldsymbol{s})-\boldsymbol{e}_i\cdot\boldsymbol{p}^*(\boldsymbol{s}) = 0$. Summing across the buyers, we get $\boldsymbol{p}^*(\boldsymbol{s})\cdot\left(\sum_{i\in[n]}\boldsymbol{x}_i^*(\boldsymbol{s}) - \sum_{i\in[n]}\boldsymbol{e}_i\right) + \boldsymbol{q}^*(\boldsymbol{s})\cdot\left(\sum_{i\in[n]}\boldsymbol{y}_i^*(\boldsymbol{s})\right) = 0$ for any $\boldsymbol{s}\in\mathcal{S}$, which is the Walras' law.

Finally, we want to show that $(\boldsymbol{X}^*, \boldsymbol{Y}^*, \boldsymbol{p}^*, \boldsymbol{q}^*)$ is feasible. We first show that $\sum_{i\in[n]}\boldsymbol{x}_i^*(\boldsymbol{s}) - \sum_{i\in[n]}\boldsymbol{e}_i \leq \boldsymbol{0}_m$ for any $\boldsymbol{s}\in\mathcal{S}$. We proved that for any state $\boldsymbol{s}\in\mathcal{S}$, $r_{n+1}'(\boldsymbol{s}, \boldsymbol{X}^*(\boldsymbol{s}), \boldsymbol{Y}^*(\boldsymbol{s}), \boldsymbol{p}^*(\boldsymbol{s}), \boldsymbol{q}^*(\boldsymbol{s})) = \boldsymbol{p}^*(\boldsymbol{s})\cdot\left(\sum_{i\in[n]}\boldsymbol{x}_i^*(\boldsymbol{s}) - \sum_{i\in[n]}\boldsymbol{e}_i\right) + \boldsymbol{q}^*(\boldsymbol{s})\cdot\left(\sum_{i\in[n]}\boldsymbol{y}_i^*(\boldsymbol{s})\right) = 0$, which implies $v_{n+1}^{\boldsymbol{\pi}^*}(\boldsymbol{s}) = 0$. For any $j\in[m]$, consider a $\boldsymbol{p}:\mathcal{S}\to\mathcal{P}$ defined by $\boldsymbol{p}(\boldsymbol{s}) = \boldsymbol{j}_j$ for all $\boldsymbol{s}\in\mathcal{S}$ and a $\boldsymbol{q}:\boldsymbol{s}\to\mathcal{Q}$ defined by $\boldsymbol{q}(\boldsymbol{s}) = \boldsymbol{0}_l$ for all $\boldsymbol{s}\in\mathcal{S}$. Then, we know that

$$
\begin{aligned}
0 &= v_{n+1}^{\boldsymbol{\pi}^*} && && (14.49)\\
&= q_{n+1}^{\boldsymbol{\pi}^*}(\boldsymbol{s}, \boldsymbol{X}^*(\boldsymbol{s}), \boldsymbol{Y}^*(\boldsymbol{s}), \boldsymbol{p}^*(\boldsymbol{s}), \boldsymbol{q}^*(\boldsymbol{s})) && && (14.50)\\
&\geq q_{n+1}^{\boldsymbol{\pi}^*}(\boldsymbol{s}, \boldsymbol{X}^*(\boldsymbol{s}), \boldsymbol{Y}^*(\boldsymbol{s}), \boldsymbol{p}(\boldsymbol{s}), \boldsymbol{q}(\boldsymbol{s})) && && (14.51)\\
&= r_{n+1}'(\boldsymbol{s}, \boldsymbol{X}^*(\boldsymbol{s}), \boldsymbol{Y}^*(\boldsymbol{s}), \boldsymbol{p}(\boldsymbol{s}), \boldsymbol{q}(\boldsymbol{s})) + \mathop{\mathbb{E}}_{S'\sim\rho(S'|\boldsymbol{s},\boldsymbol{Y}^*(\boldsymbol{s}))}[\gamma v_i^{\boldsymbol{\pi}^*}(S')] && && (14.52)\\
&= \boldsymbol{j}_j\cdot\left(\sum_{i\in[n]}\boldsymbol{x}_i^*(\boldsymbol{s}) - \sum_{i\in[n]}\boldsymbol{e}_i\right) && \forall j\in[m] && (14.53)\\
&= \sum_{i\in[n]} x_{ij}^*(\boldsymbol{s}) - \sum_{i\in[n]} e_{ij} && \forall j\in[m] && (14.54)
\end{aligned}
$$

Thus, we know that $\sum_{i\in[n]} \boldsymbol{x}_i^*(\boldsymbol{s}) - \sum_{i\in[n]} \boldsymbol{e}_i \leq \boldsymbol{0}_m$ for any $\boldsymbol{s} \in \mathcal{S}$. Finally, we show that $\sum_{i\in[n]} \boldsymbol{y}_i^*(\boldsymbol{s}) \leq \boldsymbol{0}_l$ for all $\boldsymbol{s} \in \mathcal{S}$. By way of contradiction, suppose that for some asset $k \in [l]$, and some state $\boldsymbol{s} \in \mathcal{S}$, $\sum_{i\in[n]} y_{ik}^*(\boldsymbol{s}) > 0$. Then, the auctioneer can increase its cumulative payoff by increasing $q_k^*(\boldsymbol{s})$, which contradicts the definition of a GMPE.

Therefore, we can conclude that $\boldsymbol{\pi}^* = (\boldsymbol{X}^*, \boldsymbol{Y}^*, \boldsymbol{p}^*, \boldsymbol{q}^*) : \mathcal{S} \to \mathcal{X} \times \mathcal{Y} \times \mathcal{P} \times \mathcal{Q}$ is a RRE of $\mathcal{I}$.

Finally, notice that the transition functions set in our game are all stochastically concave and as such give rise action-value functions which are concave in the actions each of player (Atakan, 2003a), and it is easy to verify that the game also satisfies all conditions that guarantee the existence of a GMPE (see Section 4 of (Atakan, 2003a) for detailed proofs). Hence, by Theorem 12.2.1 which guarantees the existence of GMPE in generalized Markov games, we can conclude that there exists an RRE $(\boldsymbol{X}^*, \boldsymbol{Y}^*, \boldsymbol{p}^*, \boldsymbol{q}^*)$ in any Radner economy $\mathcal{I}$.

14.1.3 Omitted Results and Proofs from Section 13.1.4

Theorem 13.1.2.

Consider a Radner economy $\mathcal{I}$ and the associated Radner Markov pseudo-games $\mathcal{M}$. Let $(\boldsymbol{\pi}, \boldsymbol{\rho}, \mathbb{R}^{\Omega}, \mathbb{R}^{\Sigma})$ be a parametrization scheme for $\mathcal{M}$ and suppose Assumptions 12.3.2, 12.3.3, and 13.1.1 hold. Then, the convergence results in Theorem 12.3.1 hold for $\mathcal{M}$.

Proof

This results follows readily from Theorem 13.1.1 as an application of Theorem 12.3.1.

14.2 Experiments

14.2.1 Neural Projection Method

The projection method (Judd, 1992), also known as the weighted residual methods, is a numerical technique often used to approximate solutions to complex economic models, particularly those involving dynamic programming and dynamic stochastic general equilibrium (DSGE) models. These models are common in macroeconomics and often don't have analytical solutions due to their non-linear, dynamic, and high-dimensional nature. The projection method helps approximate these solutions by projecting the problem into a more manageable, lower-dimensional space.

The main idea of the projection method is to express equilibrium of the dynamic economic model as a solution to a functional equation $D(f) = \mathbf{0}$, where $f : \mathcal{S} \to \mathbb{R}^m$ is a function that represent some unknown policy, $D : (\mathcal{S} \to \mathbb{R}^m) \to (\mathcal{S} \to \mathbb{R}^n)$, and $\mathbf{0}$ is the zero vector. Some classic examples of the operator D includes Euler equations and Bellman equations. A canonical project method consists of four steps: 1) Define a set of basis functions $\{\psi_i : \mathcal{S} \to \mathbb{R}^m\}_{i \in [n]}$ and approximate each each function $f \in \mathcal{F}$ through a linear combination of basis functions: $\hat{f}(\cdot; \boldsymbol{\theta}) = \sum_{i=1}^{n} \theta_i \psi_i(\cdot)$; 2) Define a residual equation as a functional equation evaluated at the approximation: $R(\cdot; \boldsymbol{\theta}) \doteq D(\hat{f}(\cdot; \boldsymbol{\theta}))$; 3) Choose some weight functions $\{w_i : \mathcal{S} \to \mathbb{R}\}_{i \in [p]}$ over the states and find $\boldsymbol{\theta}$ that solves $F(\boldsymbol{\theta}) \doteq \int_{\mathcal{S}} w_i(\boldsymbol{s}) R(\boldsymbol{s}; \boldsymbol{\theta}) d\boldsymbol{s} = 0$ for all $i \in [p]$. This gets the residual "close" to zero in the weighted integral sense; 4) Simulate the optimal decision rule based on the chosen parameter $\boldsymbol{\theta}$ and basis functions.

Recently, the neural projection method was developed to extend the traditional projection method (Maliar et al., 2021; Azinovic et al., 2022; Sauzet, 2021). In the neural projection method, neural networks are used as the functional approximators for policy functions instead of traditional basis function approximations. In this section, we show how we can approximate generalized Markov Perfect Nash equilibrium of generalized Markov game, and consequently Recursive Radner Equilibrium of Radner economies, through the neural projection method.

Assumption 14.2.1.

Given a generalized Markov game $\mathcal{M}$, assume that 1. for any $i \in [n]$, $\boldsymbol{s} \in \mathcal{S}$, $\boldsymbol{a}_{-i} \in \mathcal{A}_{-i}$, $\mathcal{X}_i(\boldsymbol{s}, \boldsymbol{a}_{-i}) \doteq \{\boldsymbol{a}_i \in \mathcal{A}_i \mid h_{ic}(\boldsymbol{s}, \boldsymbol{a}_i, \boldsymbol{a}_{-i}) \geq 0 \text{ for all } c \in [l]\}$ for a collection of **constraint functions** $\{h_{ic} : \mathcal{S} \times \mathcal{A} \mid c \in [l]\}$, where $\boldsymbol{a}_i \mapsto h_{ic}(\boldsymbol{s}, \boldsymbol{a}_i, \boldsymbol{a}_{-i})$ is concave for every $c \in [l]$.

Theorem 14.2.1.

Let $\mathcal{M}$ be a generalized Markov game that satisfies Assumption 14.2.1. For any policy profile $\boldsymbol{\pi} \in \mathcal{F}^{\text{markov}}$, $\boldsymbol{\pi}$ is a MPGNE if and only if there exists Lagrange multiplier policy $\boldsymbol{\lambda} : \mathcal{S} \to \mathbb{R}_+^{n \times l}$ such that $(\boldsymbol{\pi}, \boldsymbol{\lambda})$ solves the following functional equation: for all $i \in [n]$, $\boldsymbol{s} \in \mathcal{S}$,

$$0 \in \partial_{\boldsymbol{a}_i} q_i^{\boldsymbol{\pi}}(\boldsymbol{s}, \boldsymbol{\pi}_i(\boldsymbol{s}), \boldsymbol{\pi}_{-i}(\boldsymbol{s})) + \sum_{c \in [l]} \lambda_{i,c}(\boldsymbol{s}) \partial_{\boldsymbol{a}_i} h_{ic}(\boldsymbol{s}, \boldsymbol{\pi}_i(\boldsymbol{s}), \boldsymbol{\pi}_{-i}(\boldsymbol{s})) \tag{14.55}$$

$$\forall c \in [l], \quad 0 = \lambda_{ic}(\boldsymbol{s}) h_{ic}(\boldsymbol{s}, \boldsymbol{\pi}_i(\boldsymbol{s}), \boldsymbol{\pi}_{-i}(\boldsymbol{s})) \tag{14.56}$$

$$\forall c \in [l], \quad 0 \leq h_{ic}(\boldsymbol{s}, \boldsymbol{a}_i^*, \boldsymbol{\pi}_{-i}(\boldsymbol{s})) \tag{14.57}$$

$$\tag{14.58}$$

and for all $i \in [n]$, $\boldsymbol{s} \in \mathcal{S}$,

$$v_i^{\boldsymbol{\pi}}(\boldsymbol{s}) = q_i^{\boldsymbol{\pi}}(\boldsymbol{s}, \boldsymbol{\pi}_i(\boldsymbol{s}), \boldsymbol{\pi}_{-i}(\boldsymbol{s})) \tag{14.59}$$

Proof

First, we know that a policy profile $\boldsymbol{\pi} \in \mathcal{F}^{\text{markov}}$ is a MPGNE if and only if it satisfies the following generalized Bellman Optimality equations, i.e., for all $i \in [n]$, $\boldsymbol{s} \in \mathcal{S}$,

$$v_i^{\boldsymbol{\pi}}(\boldsymbol{s}) = \max_{\boldsymbol{a}_i \in \mathcal{X}_i(\boldsymbol{s}, \boldsymbol{\pi}_{-i}(\boldsymbol{s}))} r_i(\boldsymbol{s}, \boldsymbol{a}_i, \boldsymbol{\pi}_{-i}(\boldsymbol{s})) + \mathbb{E}_{\boldsymbol{s}' \sim \rho(\cdot | \boldsymbol{s}, \boldsymbol{a}_i, \boldsymbol{\pi}_{-i}(\boldsymbol{s}))}[\gamma v_i^{\boldsymbol{\pi}}(\boldsymbol{s}')] \tag{14.60}$$

$$= \max_{\boldsymbol{a}_i \in \mathcal{X}_i(\boldsymbol{s}, \boldsymbol{\pi}_{-i}(\boldsymbol{s}))} q_i^{\boldsymbol{\pi}}(\boldsymbol{s}, \boldsymbol{a}_i, \boldsymbol{\pi}_{-i}(\boldsymbol{s})) \tag{14.61}$$

Then, since $\boldsymbol{a}_i \mapsto q_i^{\boldsymbol{\pi}}(\boldsymbol{s}, \boldsymbol{a}_i, \boldsymbol{\pi}_{-i}(\boldsymbol{s}))$ is concave over $\mathcal{X}_i(\boldsymbol{s}, \boldsymbol{\pi}_{-i}(\boldsymbol{s}))$ by Assumption 12.2.1, the KKT conditions provides sufficient and necessary optimality conditions for the constrained maximization problem

$$\max_{\boldsymbol{a}_i \in \mathcal{X}_i(\boldsymbol{s}, \boldsymbol{\pi}_{-i}(\boldsymbol{s}))} q_i^{\boldsymbol{\pi}}(\boldsymbol{s}, \boldsymbol{a}_i, \boldsymbol{\pi}_{-i}(\boldsymbol{s})) \tag{14.62}$$

That is, $\boldsymbol{a}_i^* \in \mathcal{X}_i(\boldsymbol{s}, \boldsymbol{\pi}_{-i}(\boldsymbol{s}))$ is a solution to eq. (14.62) if and only if there exists $\{\lambda_{ic}^* : \mathcal{S} \to \mathbb{R}_+\}_{c \in [l]}$ s.t.

$$0 \in \partial_{\boldsymbol{a}_i} q_i^{\boldsymbol{\pi}}(\boldsymbol{s}, \boldsymbol{a}_i^*, \boldsymbol{\pi}_{-i}(\boldsymbol{s})) + \sum_{c \in [l]} \lambda_{ic}^*(\boldsymbol{s}) \partial_{\boldsymbol{a}_i} h_{ic}(\boldsymbol{s}, \boldsymbol{a}_i^*, \boldsymbol{\pi}_{-i}(\boldsymbol{s})) \tag{14.63}$$

$$\forall c \in [l], \quad 0 = \lambda_{ic}^*(\boldsymbol{s}) h_{ic}(\boldsymbol{s}, \boldsymbol{a}_i^*, \boldsymbol{\pi}_{-i}(\boldsymbol{s})) \tag{14.64}$$

$$\forall c \in [l], \quad 0 \leq h_{ic}(\boldsymbol{s}, \boldsymbol{a}_i^*, \boldsymbol{\pi}_{-i}(\boldsymbol{s})) \tag{14.65}$$

Therefore, we can conclude that $\boldsymbol{\pi} \in \mathcal{F}^{\text{markov}}$ is a MPGNE if and only if there exists $\{\lambda_{ic} : \mathcal{S} \to \mathbb{R}_+\}_{i\in[n],c\in[l]}$ s.t. for all $i \in [n]$, $\boldsymbol{s} \in \mathcal{S}$,

$$0 \in \partial_{\boldsymbol{a}_i} q_i^{\boldsymbol{\pi}}(\boldsymbol{s}, \boldsymbol{\pi}_i(\boldsymbol{s}), \boldsymbol{\pi}_{-i}(\boldsymbol{s})) + \sum_{c\in[l]} \lambda_{i,c}(\boldsymbol{s}) \partial_{\boldsymbol{a}_i} h_{ic}(\boldsymbol{s}, \boldsymbol{\pi}_i(\boldsymbol{s}), \boldsymbol{\pi}_i(\boldsymbol{s})) \tag{14.66}$$

$$\forall c \in [l], \quad 0 = \lambda_{ic}(\boldsymbol{s}) h_{ic}(\boldsymbol{s}, \boldsymbol{\pi}_i(\boldsymbol{s}), \boldsymbol{\pi}_i(\boldsymbol{s})) \tag{14.67}$$

$$\forall c \in [l], \quad 0 \le h_{ic}(\boldsymbol{s}, \boldsymbol{a}_i^*, \boldsymbol{\pi}_{-i}(\boldsymbol{s})) \tag{14.68}$$

and for all $i \in [n]$, $\boldsymbol{s} \in \mathcal{S}$,

$$v_i^{\boldsymbol{\pi}}(\boldsymbol{s}) = q_i^{\boldsymbol{\pi}}(\boldsymbol{s}, \boldsymbol{\pi}_i(\boldsymbol{s}), \boldsymbol{\pi}_{-i}(\boldsymbol{s})) \tag{14.69}$$

Therefore, for a policy profile $\boldsymbol{\pi} \in \mathcal{F}^{\text{markov}}$ and a Lagrange multiplier policy $\boldsymbol{\lambda} : \mathcal{S} \to \mathbb{R}_+^{n\times l}$ such that $(\boldsymbol{\pi}, \boldsymbol{\lambda})$, consider the **total first order violation**

$$\Xi_{\text{first-order}}(\boldsymbol{\pi}, \boldsymbol{\lambda}) = \sum_{i\in[n]} \left\| \int_{\boldsymbol{s}\in\mathcal{S}} \partial_{\boldsymbol{a}_i} q_i^{\boldsymbol{\pi}}(\boldsymbol{s}, \boldsymbol{\pi}_i(\boldsymbol{s}), \boldsymbol{\pi}_{-i}(\boldsymbol{s})) + \sum_{c\in[l]} \lambda_{i,c}(\boldsymbol{s}) \partial_{\boldsymbol{a}_i} h_{ic}(\boldsymbol{s}, \boldsymbol{\pi}_i(\boldsymbol{s}), \boldsymbol{\pi}_{-i}(\boldsymbol{s})) ds \right\|_2^2 \tag{14.70}$$

and the **average Bellman error**

$$\Xi_{\text{Bellman}}(\boldsymbol{\pi}, \boldsymbol{\lambda}) = \sum_{i\in[n]} \left\| \int_{\boldsymbol{s}\in\mathcal{S}} v_i^{\boldsymbol{\pi}}(\boldsymbol{s}) - q_i^{\boldsymbol{\pi}}(\boldsymbol{s}, \boldsymbol{\pi}_i(\boldsymbol{s}), \boldsymbol{\pi}_{-i}(\boldsymbol{s})) ds \right\|_2^2. \tag{14.71}$$

We can directly approximate the MPGNE through minimizing the sum of these two errors.

Typically, approximating the MPGNE using the neural projection method requires optimizing both the policy profile and the Lagrange multiplier policy. However, in exchange economy Markov pseudo-games, we derive a closed-form solution for the optimal Lagrange multiplier, allowing us to focus solely on optimizing the policy profile.

14.2.2 Implementation Details

Deterministic Case Training Details For deterministic transition probability case, for each reward function class we randomly sampled one economy with 10 consumers, 10 commodities, 1 asset, and 5 world state. The asset return matrix $\boldsymbol{R}$ is sampled in a way such that $r_{okj} \sim \text{Unif}([0.5, 1.1])$ for all o, k, and j. Moreover, we set the length of the stochastic process to be 30. For the initial state, we sample each consumer's endowment $\boldsymbol{e}_i \sim \text{Unif}([0.01, 0.1])^m$ and normalized so that the total endowment of each commodity add up to 1. We also sample each consumer's type $\boldsymbol{\theta}_i \sim \text{Unif}([1.0, 5.0])^m$, and set the world state to be 0. The transition probability function ρ is defined as $\rho(\boldsymbol{s}' \mid \boldsymbol{s}, \boldsymbol{Y}) = 1$ for all $\boldsymbol{s}(o, \boldsymbol{E}, \boldsymbol{\Theta})$ where $\boldsymbol{s}' = (o', \boldsymbol{E}', \boldsymbol{\Theta}')$ is defined as $o' = 0$, $\boldsymbol{E}' = 0.01 \cdot \mathbf{1}_{n \times m}$, and $\boldsymbol{\Theta}' = \boldsymbol{\Theta}$.

Then, for both GAPNets method and neural projection method, we run 1000 episodes for each learning rate candidate in a grid search manner and measure the performance in terms of minimizing total first-order violation and average Bellman error. Finally, we pick the best hyperparameter for the final experiments.

In the final experiments, we run GAPNets for 2000 episodes using learning rates $\boldsymbol{\eta}_{\boldsymbol{\omega}} = 1 \times 10^{-5}, \boldsymbol{\eta}_{\boldsymbol{\sigma}} = 1 \times 10^{-5}$ for the linear economy, $\boldsymbol{\eta}_{\boldsymbol{\omega}} = 1 \times 10^{-5}, \boldsymbol{\eta}_{\boldsymbol{\sigma}} = 1 \times 10^{-5}$ for the Cobb-Douglas economy, and $\boldsymbol{\eta}_{\boldsymbol{\omega}} = 1 \times 10^{-5}, \boldsymbol{\eta}_{\boldsymbol{\sigma}} = 1 \times 10^{-5}$ for the Leontief economy. Similarly, we ran neural projection method for 2000 episodes using learning rates $\boldsymbol{\eta}_{\boldsymbol{\omega}} = 1 \times 10^{-4}$ for the linear economy, $\boldsymbol{\eta}_{\boldsymbol{\omega}} = 2.5 \times 10^{-5}$ for the Cobb-Douglas economy, and $\boldsymbol{\eta}_{\boldsymbol{\omega}} = 1 \times 10^{-4}$ for the Leontief economy. In this process, we compute the exploitability of computed policy profile through gradient ascent of the adversarial network. In specific, we ran 1000 episodes of gradient ascent with learning rate $\boldsymbol{\eta}_{\boldsymbol{\sigma}} = 5 \times 10^{-5}$ for the linear economy, $\boldsymbol{\eta}_{\boldsymbol{\sigma}} = 1 \times 10^{-4}$ for the Cobb-Douglas economy, and $\boldsymbol{\eta}_{\boldsymbol{\sigma}} = 1 \times 10^{-4}$ for the Leontief economy.

Next, for each economy, we randomly sample 50 policy profiles and record their total first-order violations, average Bellman errors, and exploitabilities. Finally, we normalize the results by the average of the sampled values.

Stochastic Case Training Details For stochastic transition probability case, for each reward function class we randomly sampled one economy with 10 consumers, 10 commodities, 1 asset, and 5 world state. The asset return matrix $\boldsymbol{R}$ is sampled in a way such that $r_{okj} \sim \text{Unif}([0.5, 1.1])$ for all o, k, and j. Moreover, we set the length of the stochastic process to be 30. For the initial state, we sample each consumer's endowment $\boldsymbol{e}_i \sim \text{Unif}([0.01, 0.1])^m$ and normalized so that the total endowment of each commodity add up to 1. We also sample each consumer's type $\boldsymbol{\theta}_i \sim \text{Unif}([1.0, 5.0])^m$, and set the world state to be 0. The transition probability function will stochastically transition from state $\boldsymbol{s}(o, \boldsymbol{E}, \boldsymbol{\Theta})$ to state $\boldsymbol{s}' = (o', \boldsymbol{E}', \boldsymbol{\Theta}')$ where $o' \sim \text{Unif}(\{0, 1, 2, 3, 4\})$, $\boldsymbol{E}' \sim 0.002 + \text{Unif}([0.01, 0.1])^{n \times m}$, and $\boldsymbol{\Theta}' = \boldsymbol{\Theta}$.

Then, for both GAPNets method and neural projection method, we run 1000 episodes for each learning rate candidate in a grid search manner and measure the performance in terms of minimizing total first-order violation and average Bellman error. Finally, we pick the best hyperparameter for the final experiments.

In the final experiments, we run GAPNets for 2000 episodes using learning rates $\eta_\omega = 1 \times 10^{-5}, \eta_\sigma = 1 \times 10^{-5}$ for the linear economy, $\eta_\omega = 2.5 \times 10^{-5}, \eta_\sigma = 2.5 \times 10^{-5}$ for the Cobb-Douglas economy, and $\eta_\omega = 5 \times 10^{-5}, \eta_\sigma = 5 \times 10^{-5}$ for the Leontief economy. Similarly, we ran neural projection method for 2000 episodes using learning rates $\eta_\omega = 5 \times 10^{-5}$ for the linear economy, $\eta_\omega = 2.5 \times 10^{-5}$ for the Cobb-Douglas economy, and $\eta_\omega = 5 \times 10^{-4}$ for the Leontief economy. In this process, we compute the exploitability of computed policy profile through gradient ascent of the adversarial network. In specific, we ran 1000 episodes of gradient ascent with learning rate $\eta_\sigma = 7.5 \times 10^{-4}$ for the linear economy, $\eta_\sigma = 1 \times 10^{-4}$ for the Cobb-Douglas economy, and $\eta_\sigma = 1 \times 10^{-4}$ for the Leontief economy. When estimating the neural loss function—cumulative regret for the GAPNets method and total first-order violations and average Bellman error for the neural projection method—we use 100 samples for GAPNets and 10 samples for the neural projection method. The primary reason for this difference is the high memory consumption of the neural projection method, which makes larger sample sizes infeasible.

Next, for each economy, we randomly sample 50 policy profiles and record their total first-order violations, average Bellman errors, and exploitabilities. Finally, we normalize the results by the average of the sampled values.

14.2.3 Other Details

Programming Languages, Packages, and Licensing We ran our experiments in Python 3.7 (Van Rossum and Drake Jr, 1995), using NumPy (Harris et al., 2020), , CVXPY (Diamond and Boyd, 2016), Jax (Bradbury et al., 2018), OPTAX (Bradbury et al., 2018), Haiku (Hennigan et al., 2020), and JaxOPT (Blondel et al., 2021). All figures were graphed using Matplotlib (Hunter, 2007).

Python software and documentation are licensed under the PSF License Agreement. Numpy is distributed under a liberal BSD license. Pandas is distributed under a new BSD license. Matplotlib only uses BSD compatible code, and its license is based on the PSF license. CVXPY is licensed under an APACHE license.

Computational Resources The experiments were conducted using Google Colab, which provides cloud-based computational resources. Specifically, we utilized an NVIDIA T4 GPU with the following specifications: GPU: NVIDIA T4 (16GB GDDR6), CPU: Intel Xeon (2 vCPUs), RAM: 12GB, Storage: Colab-provided ephemeral storage.

Code Repository the full details of our experiments, including hyperparameter search, final experiment configurations, and visualization code, can be found in our code repository (`https://anonymous.4open.science/r/Markov-Pseudo-Game-EC2025-DCB8`).

Chapter 15

Conclusion

This thesis addresses the computational challenge of solving general equilibrium models by integrating techniques from computer science, optimization, and game theory. For over half a century, researchers have sought a general numerical method to compute equilibria in complex economies, a pursuit that began with Herbert Scarf's foundational work (Scarf, 1960) which itself was inspired by the works of Walras (1896), Uzawa (1960), and Arrow and Debreu (1954). While prior methods achieved partial success in small-scale models, no comprehensive solution existed for large, realistic economic systems. This thesis advances the field by introducing novel optimization frameworks, theoretical guarantees, and practical algorithms, offering a systematic resolution to this long-standing problem.

The first part of the thesis introduces variational inequalities (VIs) as a mathematical foundation for modeling Walrasian economies. A new class of first-order methods, the mirror extragradient algorithm, is developed, achieving polynomial-time convergence under the Minty condition and Bregman continuity. These results also include the first local convergence guarantee for constrained Bregman-continuous VIs without the Minty condition. This theoretical groundwork enables the formulation of the mirror *extratâtonnement* process, a price-adjustment mechanism that converges in all balanced Walrasian economies. This process provides the first known polynomial-time, globally convergent price-adjustment method for solving Walrasian equilibria, resolving a major computational challenge in general equilibrium theory. Additionally, this thesis analyzes the convergence of *tâtonnement* in homothetic Fisher markets, offering a unified understanding of *tâtonnement* behavior across different utility functions.

Building on these results, the second part of the thesis extends the analysis to pseudo-games, a generalization of multiagent optimization frameworks, and their application to Arrow-Debreu economies. A new family of uncoupled learning dynamics, called mirror extragradient learning, is introduced, providing polynomial-time convergence guarantees for variationally stable pseudo-games. This part then reframes Arrow-Debreu equilibria as solutions to pseudo-games, enabling a computationally efficient characterization of equilibria in pure exchange economies through the application of mirror extragradient learning in the variationally stable trading post pseudo-game. This marks a significant advance in the computation of Arrow-Debreu equilibrium in pure exchange economies, for which no globally convergent market dynamics were known.

The final part of this thesis explores Markov pseudo-games, extending the previous frameworks to Radner economies, which explicitly incorporate time and uncertainty. A novel learning-based approach is introduced for computing generalized Markov perfect equilibria (GMPE), leveraging adversarial learning techniques to compute equilibrium policies in polynomial time. This section also extends pure equilibrium existence results in Markov games to settings with continuous action spaces, where previously only mixed-strategy equilibria were known. This part then focuses on the computation of Radner equilibria, an inherently infinite-dimensional problem. A function-approximation method inspired by merit functions is introduced, allowing for efficient computation of a solution which satisfies necessary conditions to be a Radner equilibrium under suitable smoothness conditions. These findings open a new research direction at the intersection of deep learning, reinforcement learning, and mathematical economics, offering a promising path toward scalable, data-driven economic models.

The ordering of the three major parts is intentional and serves three key purposes. First, the results on VIs form the mathematical backbone of the thesis and are used throughout, requiring them to be presented first. Second, the fact that Walrasian economies can be seen as a special case of Arrow-Debreu economies suggests a natural progression from one to the other, with insights from Walrasian models helping to contextualize results in Arrow-Debreu markets. Finally, Markov pseudo-games and Radner economies extend the previous models to infinite-dimensional settings, where equilibrium solutions become significantly more complex and require modern learning-based computational approaches. This natural progression not only unifies classical and modern equilibrium models but also provides a clear research trajectory toward infinite-dimensional optimization and economies with infinitely many commodities, a rapidly growing field in economic theory and applied mathematics.

In summary, this thesis provides a unified computational framework for solving general equilibrium models, integrating classical economic theory with modern optimization and learning techniques. It resolves long-standing computational challenges in computing Walrasian, Arrow-Debreu, and Radner equilibria, while also introducing novel algorithmic tools that have broader implications for optimization, game theory, and artificial intelligence.

15.1 Future Directions

I now highlight several promising directions for future research that I find both exciting and relevant to the public good. Rather than following the chronological order in which they were presented in this thesis, I have organized them based on their potential to advance research and enhance the application of general equilibrium theory in real-world settings.

15.1.1 Radner Economies and Infinite-Dimensional Walrasian Economies

The Walrasian and Arrow-Debreu economy models studied in this thesis were finite-dimensional in that they considered only a finite set of commodities. More recently, infinite-dimensional generalizations of these economies have been explored (Prescott and Lucas, 1972), and equilibrium existence has been established. However, apart from one notable work (Gao and Kroer, 2021), little is known about computing Walrasian equilibria in infinite-dimensional settings. Advancing this research is crucial for applying general equilibrium models to policy analysis, as macroeconomic policy models often take the form of Radner economies, which themselves can be viewed as special cases of infinite-dimensional Walrasian economies. Specifically, any Radner economy can be reformulated as a Walrasian economy in which the set of goods is given by the union of all commodities and assets across all states. Since the state space in Radner economies is typically continuous, the resulting Walrasian economy is generally infinite-dimensional. Indeed, the purpose of the study of Radner economies in this thesis was to push the envelope of algorithmic general equilibrium towards infinite dimensional economies.

In Part III, I propose one approach to solving infinite-dimensional economies, but this merely scratches the surface of what is possible. Given the complexity of these problems, machine learning and artificial intelligence are likely to play a pivotal role in their resolution. Future research should focus on developing methods for even more complex stochastic and high-dimensional economies, incorporating deep learning techniques to improve scalability and robustness. This emerging field, at the intersection of deep learning, reinforcement learning, optimization theory, and mathematical economics, holds immense potential. More importantly, it offers a long-overdue opportunity to bring the transformative power of AI to economic policy-making—an area that has remained largely untouched by these advances for far too long.

15.1.2 Walrasian Economies

The theoretical insights and empirical validation presented in Part I suggest that the computational intractability of general equilibrium problems arises primarily from discontinuities rather than inherent complexity. This perspective challenges long-standing assumptions in applied general equilibrium theory, particularly those stemming from the works of Scarf (1960), Papadimitriou and Yannakakis (2010), Codenotti et al. (2006), and Deng and Du (2008). Our empirical results indicate that the mirror *extratâtonnement* process can compute Walrasian equilibria in cases where prior theoretical results suggested this would be infeasible (e.g., Leontief Arrow-Debreu economies (Codenotti et al., 2006; Deng and Du, 2008)). While I have attempted to explain these observations through the pathwise Bregman-smoothness assumption, future work should explore specific kernel functions and characterize the class of Walrasian economies where this assumption holds as past work has done (Goktas et al., 2023b). Doing so would provide deeper insights into the computational challenges identified over the past two decades.

Moreover, our experiments demonstrate the ability of the mirror *extratâtonnement* process to solve very large Leontief Arrow-Debreu economies. Since these economies are known to be PPAD-complete (Deng and Du, 2008; Codenotti et al., 2006), there exists a polynomial-time reduction from games to Leontief Arrow-Debreu economies. This suggests that the mirror *extratâtonnement* process could also be used to solve large games in practice. Future research should investigate such algorithms, potentially exploring formulations of Nash equilibria as discontinuous variational inequalities satisfying the Minty condition.

15.1.3 Arrow-Debreu Economies

While the trading post pseudo-game provides a tractable characterization of Arrow-Debreu equilibria in pure exchange economies, it remains an open question whether this approach extends to concave Arrow-Debreu economies. Future work should explore whether the trading post pseudo-game can be generalized to characterize all Arrow-Debreu economies, thereby paving the way for market dynamics that converge to equilibrium in a broader class of Arrow-Debreu economies.

15.1.4 Fisher Markets

Future work could investigate the space of homogeneous utility functions with negative cross-price elasticity of Hicksian demand to possibly derive faster convergence rates than those provided in this paper. Additionally, it remains to be seen if the bound we have provided in this paper is tight; the greatest lower bound known for the convergence of *tâtonnement* in homothetic Fisher markets is $O(1/t^2)$ for Leontief markets (Cheung et al., 2012), leaving space for improvement. Finally, Lemma 6.4.1 (Chapter 6) suggests that to extend convergence results for *tâtonnement* beyond homothetic domains, one might have to consider the Hicksian demand elasticity *w.r.t. utility level* rather than price.

15.1.5 Variational Inequalities

The polynomial-time best-iterate convergence of the mirror extragradient method to an approximate strong solution has been established under the assumption that the kernel function is strongly convex and Lipschitz-smooth. However, it is likely that the Lipschitz-smoothness assumption can be dropped, as it does not appear necessary to show that the distance between intermediate and final iterates decreases. This generalization would be highly useful, given that many Bregman divergences are generated by strongly convex functions that are not Lipschitz-smooth (e.g., Itakura-Saito Divergence (Itakura and Saito, 1968)). Another promising direction is to investigate whether the mirror extragradient method can be shown to converge in polynomial time to an approximate strong solution in last iterates under the Minty condition.

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